# Pricing Rainbow Options

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**Abstract**

A previous paper (West 2005) tackled the issue of calculating accurate uni-, bi- and trivariate normal probabilities. This has important applications in the pricing of multi-asset options, e.g. rainbow options. In this paper, we derive the Black—Scholes prices of several styles of (multi-asset) rainbow options using change-of-numeraire machinery. Hedging issues and deviations from the Black-Scholes pricing model are also briefly considered.

**Keywords**

exotic option, Black-Scholes model, exchange option, rainbow option, equivalent martingale measure, change of numeraire, trivariate normal.

## 1. Definition of a Rainbow Option

Rainbow Options refer to all options whose payoff depends on more than one underlying risky asset; each asset is referred to as a colour of the rainbow. Examples of these include:

- "Best of assets or cash" option, delivering the maximum of two risky assets and cash at expiry (Stulz 1982), (Johnson 1987), (Rubinstein 1991)
- "Call on max" option, giving the holder the right to purchase the maximum asset at the strike price at expriry, (Stulz 1982), (Johnson 1987)
- "Call on min" option, giving the holder the right to purchase the minimum asset at the strike price at expiry (Stulz 1982), (Johnson 1987)
- "Put on max" option, giving the holder the right to sell the maximum of the risky assets at the strike price at expiry, (Margrabe 1978), (Stulz 1982), (Johnson 1987)
- "Put on min" option, giving the holder the right to sell the minimum of the risky assets at the strike at expiry (Stulz 1982), (Johnson 1987)
- "Put 2 and call 1", an exchange option to put a predefined risky asset and call the other risky asset, (Margrabe 1978). Thus, asset 1 is called with the 'strike' being asset 2.

Thus, the payoffs at expiry for rainbow European options are:

| | |
|---|---|
| Best of assets or cash | $\max(S_1, S_2, \ldots, S_n, K)$ |
| Call on max | $\max(\max(S_1, S_2, \ldots, S_n) - K, 0)$ |
| Call on min | $\max(\min(S_1, S_2, \ldots, S_n) - K, 0)$ |
| Put on max | $\max(K - \max(S_1, S_2, \ldots, S_n), 0)$ |
| Put on min | $\max(K - \min(S_1, S_2, \ldots, S_n), 0)$ |
| Put 2 and Call 1 | $\max(S_1 - S_2, 0)$ |

To be true to history, we deal with the last case first.

## 2. Notation and Setting

Define the following variables:

- $S_i$ = Spot price of asset $i$,
- $K$ = Strike price of the rainbow option,

- $\sigma_i$ = volatility of asset $i$,
- $q_i$ = dividend yield of asset $i$,
- $\rho_{ij}$ = correlation coefficient of return on assets $i$ and $j$,
- $r$ = the risk-free rate (NACC),
- $\tau$ = the term to expiry of the rainbow option.

Our system for the asset dynamics will be

$$\underline{dS/S} = \underline{(r-q)}dt + A\,\underline{dW} \tag{1}$$

where the Brownian motions are independent. $A$ is a square root of the covariance matrix $\Sigma$, that is $AA' = \Sigma$. As such, $A$ is not uniquely determined, but it would be typical to take A to be the Choleski decomposition matrix of $\Sigma$ (that is, $A$ is lower triangular). Under such a condition, $A$ is uniquely determined.

Let the $i^{th}$ row of $A$ be $\underline{a}_i$. We will say that $\underline{a}_i$ is the volatility vector for asset $S_i$. Note that if we were to write things where $S_i$ had a single volatility $\sigma_i$ then $\sigma_i^2 = \sum_{j=1}^n a_{ij}^2$, so $\sigma_i = \|\underline{a}_i\|$, where the norm is the usual Euclidean norm. Also, the correlation between the returns of $S_i$ and $S_j$ is given by $\frac{\underline{a}_i \cdot \underline{a}_j}{\|\underline{a}_i\|\|\underline{a}_j\|}$.

## 3. The Result of Margrabe

The theory of rainbow options starts with (Margrabe 1978) and has its most significant other development in (Stulz 1982).

(Margrabe 1978) began by evaluating the option to exchange one asset for the other at expiry. This is justifiably one of the most famous early option pricing papers. This is conceptually like a call on the asset we are going to receive, but where the strike is itself stochastic, and is in fact the second asset. The payoff at expiry for this European option is:

$$\max(S_1 - S_2, 0),$$

which can be valued as:

$$V_M = S_1 e^{-q_1\tau} N(d_+) - S_2 e^{-q_2\tau} N(d_-), \tag{2}$$

where

$$d_\pm = \frac{\ln \frac{f_1}{f_2} \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \tag{3}$$

$$f_i = S_i e^{(r-q_i)\tau} \tag{4}$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \tag{5}$$

Margrabe derives this formula by developing and then solving a Black-Scholes type differential equation. But he also gives another argument, which he credits to Stephen Ross, which with the hindsight of modern technology, would be considered to be the most appropriate approach to the problem. Let asset 2 be the numeraire in the market. In other words, asset 2 forms a new currency, and asset one costs $\frac{S_1}{S_2}$ in that currency. The risk free rate in this market is $q_2$. Thus we have the option to buy asset one for a strike of 1. This has a Black-Scholes price of

$$V = \frac{S_1}{S_2} e^{-q_1\tau} N(d_+) - e^{-q_2\tau} N(d_-)$$

$$d_\pm = \frac{\ln \frac{\frac{S_1}{S_2}}{1} + \left(q_2 - q_1 \pm \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

where $\sigma$ is the volatility of $\frac{S_1}{S_2}$. To get from a price in the new asset 2 currency to a price in the original economy, we multiply by $S_2$: the 'exchange rate', which gives us (2).

So, what is $\sigma$? We show that (5) is the correct answer to this question in §4.

## 4. Change of Numeraire

Suppose that $X$ is a European—style derivative with expiry date $T$. Since (Harrison & Pliska 1981) it has been known that if $X$ can be perfectly hedged (i.e. if there is a self—financing portfolio of underlying instruments which perfectly replicates the payoff of the derivative at expiry), then the time—$t$ value of the derivative is given by the following *risk—neutral valuation formula*:

$$X_t = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[X_T]$$

where $r$ is the riskless rate, and the symbol $\mathbb{E}_t^{\mathbb{Q}}$ denotes the expectation at time $t$ under a *risk—neutral measure* $\mathbb{Q}$. A measure $\mathbb{Q}$ is said to be risk—neutral if all discounted asset prices $\bar{S}_t = e^{-rt}S_t$ are *martingales* under the measure $\mathbb{Q}$, i.e. if the expected value of each $\bar{S}_t$ at an earlier time $u$ is its current value $\bar{S}_u$:

$$\mathbb{E}_u^{\mathbb{Q}}[\bar{S}_t] = \bar{S}_u \qquad \text{whenever } 0 \le u \le t$$

(Here we assume for the moment that S pays no dividends.)

Now let $A_t = e^{rt}$ denote the bank account. Then the above can be rewritten as

$$\frac{X_t}{A_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{X_T}{A_T}\right] \qquad \text{i.e.} \qquad \bar{X}_t = \mathbb{E}_t^{\mathbb{Q}}\left[\bar{X}_T\right]$$

Thus $\bar{X}_t$ is a $\mathbb{Q}$–martingale.

In an important paper, (Geman, El Karoui & Rochet 1995) it was shown that there is "nothing special" about the bank account: given an asset (1995) $\hat{A}$, we can "discount" each underlying asset using $\hat{A}$:

$$\hat{S}_t = \frac{S_t}{\hat{A}_t}$$

Thus $\hat{S}$ is the "price" of $S$ measured not in money, but in units of $\hat{A}$. The asset $\hat{A}$ is referred to as a numéraire, and might be a portfolio or a derivative—the only restriction is that its value $\hat{A}_t$ is strictly positive during the time period under consideration.

It can be shown (cf. (Geman, et al. 1995)) that in the absence of arbitrage, and modulo some technical conditions, there is for each numéraire (1995) $\hat{A}$ a measure $\hat{\mathbb{Q}}$ with the property that each numéraire—deflated asset price process $\hat{S}_t$ is a $\hat{\mathbb{Q}}$—martingale, i.e.

$$\mathbb{E}_u^{\hat{\mathbb{Q}}}[\hat{S}_t] = \hat{S}_u \qquad \text{whenever } 0 \le u \le t$$

(Again, we assume that $S$ pays no dividends.) We call $\hat{\mathbb{Q}}$ the equivalent martingale measure (EMM) associated with the numéraire $\hat{A}$. It then follows easily that if a European—style derivative $X$ can be perfectly hedged, then

$$\hat{X}_t = \mathbb{E}_t^{\hat{\mathbb{Q}}}[\hat{X}_T] \qquad \text{and so} \qquad X_t = \hat{A}_t \mathbb{E}_t^{\hat{\mathbb{Q}}}\left[\frac{X_T}{\hat{A}_T}\right]$$

Indeed, if $V_t$ is the value of a replicating portfolio, then (1) $X_t = V_t$ by the law of one price, and (2) $\hat{V}_t = \frac{V_t}{\hat{A}_t}$ is a $\hat{\mathbb{Q}}$—martingale. Thus

$$\hat{X}_t = \hat{V}_t = \mathbb{E}_t^{\hat{\mathbb{Q}}}\left[\hat{V}_T\right] = \mathbb{E}_t^{\hat{\mathbb{Q}}}\left[\hat{X}_T\right]$$

using the fact that $V_T = X_T$—by definition of “replicating portfolio”.

It follows that if $N_1, N_2$ are numéraires, with associated EMM's $\mathbb{Q}_1, \mathbb{Q}_2$, then

$$N_1(t)\mathbb{E}_t^{\mathbb{Q}_1}\left[\frac{X_T}{N_1(T)}\right] = N_2(t)\mathbb{E}_t^{\mathbb{Q}_2}\left[\frac{X_T}{N_2(T)}\right]$$

Indeed, both sides of the above equation are equal to the time—t price of the derivative.

To get slightly more technical, the EMM $\hat{\mathbb{Q}}$ associated with numéraire $\hat{A}$ is obtained from the risk—neutral measure $\mathbb{Q}$ via a Girsanov transformation (whose kernel is the volatility vector of the numéraire). In particular, the volatility vectors of all assets are the same under both $\mathbb{Q}$ and $\hat{\mathbb{Q}}$.

A minor modification of the above reasoning is necessary in case the assets pay dividends. Suppose that $S$ is a share with dividend yield $q$. If we buy one share at time $t = 0$, and if we reinvest the dividends in the share, we will have $e^{qt}$ shares at time $t$, with value $S(t)e^{qt}$. If $\hat{A}$ is the new numéraire, with dividend yield $\hat{q}$, then it is the ratio

$$\frac{S(t)e^{qt}}{\hat{A}(t)e^{\hat{q}t}}$$

that is a $\hat{\mathbb{Q}}$—martingale, and not the ratio $\frac{S(t)}{\hat{A}(t)}$.

Suppose now that we have $n$ assets $S_1, S_2, \ldots, S_n$, and that we model the asset dynamics using an $n$—dimensional standard Brownian motion. If $\underline{a}_i$ is the volatility vector of $S_i$, then, under the risk—neutral measure $\mathbb{Q}$, the dynamics of $S_i$ are given by

$$\frac{dS_i}{S_i} = (r - q_i)\ dt + \underline{a}_i \cdot dW$$

where $q_i$ is the dividend yield of $S_i$, and $W$ is an $n$—dimensional standard $\mathbb{Q}$—Brownian motion. When we work with asset $S_j$ as numéraire, we will be interested in the dynamics of the asset ratio processes

$$S_{i/j}(t) = \frac{S_i(t)}{S_j(t)}$$

under the associated EMM $\mathbb{Q}_j$. Now by Ito's formula the *risk—neutral* dynamics of $S_{i/j}$ are given by

$$\frac{dS_{i/j}}{S_{i/j}} = \left(q_j - q_i + \|\underline{a}_j\|^2 - \underline{a}_i \cdot \underline{a}_j\right)\ dt + (\underline{a}_i - \underline{a}_j) \cdot dW$$

However, when we change to measure $\mathbb{Q}_j$, we know that $Y(t) = S_{i/j}(t) e^{(q_i - q_j)t}$ is a $\mathbb{Q}_j$—martingale. Applying Ito's formula again, we see that the risk—neutral dynamics of $Y_t$ are given by

$$\frac{dY}{Y} = \left(\|\underline{a}_j\|^2 - \underline{a}_i \cdot \underline{a}_j\right)\ dt + (\underline{a}_i - \underline{a}_j) \cdot dW$$

Since $Y(t)$ is a $\mathbb{Q}_j$—martingale, its drift under $\mathbb{Q}_j$ is zero, and its volatility remains unchanged. Thus the $\mathbb{Q}_j$—dynamics of $Y(t)$ are

$$\frac{dY}{Y} = (\underline{a}_i - \underline{a}_j) \cdot dW^j$$

where $W^j$ is a standard $n$—dimensional $\mathbb{Q}_j$—Brownian motion. Applying Ito's formula once again to $S_{i/j}(t) = Y(t)e^{-(q_i - q_j)t}$, it follows easily that the $\mathbb{Q}_j$—dynamics of $S_{i/j}(t)$ are given by

$$\frac{dS_{i/j}}{S_{i/j}} = (q_j - q_i)\ dt + (\underline{a}_i - \underline{a}_j) \cdot dW^j$$

Returning to §3, we have $\sigma^2 = \|\underline{a}_1 - \underline{a}_2\|^2 = \|\underline{a}_1\|^2 + \|\underline{a}_2\|^2 - 2\rho\|\underline{a}_1\| \|\underline{a}_2\| = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$, as required.

## 5. The Results of Stulz

(Stulz 1982) derives the value of what are now called two asset rainbow options. First the value of the call on the minimum of the two assets is derived, by evaluating the (rather unpleasant) bivariate integral. Then a min-max parity argument is invoked: having a two asset rainbow maximum call and the corresponding two asset rainbow minimum call is just the same as having two vanilla calls on the two assets.

Finally put-call parity results are derived, enabling evaluation of the put on the minimum and the put on the maximum. Rather than going into any details we immediately proceed to the more general case where we derive far more pleasant ways of immediately finding any such valuation.

## 6. Many Asset Rainbow Options

In (Johnson 1987) extensions of the results of (Stulz 1982) are claimed to any number of underlyings. However, the formulae in the paper are actually quite difficult to interpret without ambiguity: they are presented inductively, and the formula (even for $n = 3$) is difficult to interpret with certainty. Moreover, the formulae are not proved—only intuitions are provided—nor is any numerical work undertaken to provide some comfort in the results. The arguments basically involve intuiting what the delta's of the option in each of the $n$ underlyings should be, and extrapolating from there to the price. So one can say ‘bravo’ given that it is possible to actually formally derive proofs for these many asset pricing formulae.

What we do is construct general Martingale-style arguments for all cases $n \geq 2$ which are in the style of the proof first found by Margrabe and Ross.

Johnson's results are stated for any number of assets. A rainbow option with $n$ assets will require the $n$-variate cumulative normal function for application of his formulae. As $n$ increases, so the computational effort and execution time for having such an approximation will increase

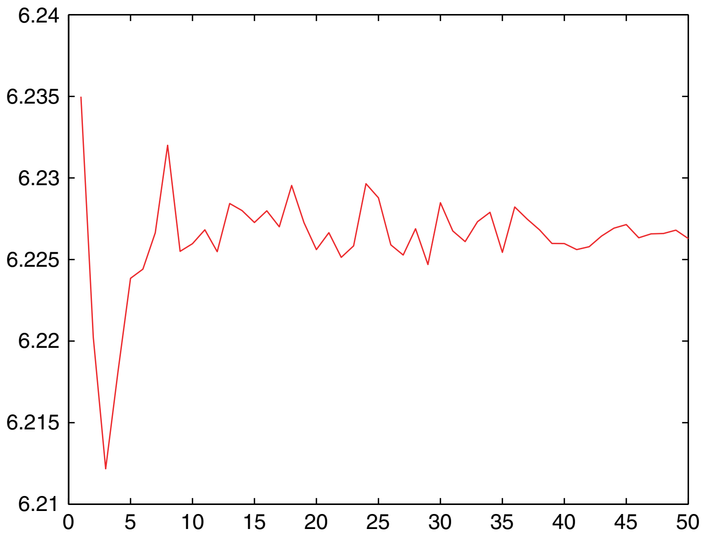


**Figure 1: Monte Carlo for call on minimum on 3 assets. On the horizontal axis: number of experiments in 1000's, using independent Sobol sequences, on the vertical axis: price. The exact option value using the formula presented here is 6.2273.**

dramatically. In (West 2005) we have vb and c++ code for $n \leq 3$ based upon the Fortran of (Genz 2004), so here we apply this code to European rainbow options with three stock underlyings, $S_1$, $S_2$ and $S_3$. Code for $n > 3$ does not seem to be available (in any language), at least in a form that would make the computational time better than direct Monte Carlo valuation of the original option.

Using that code, for the case $n = 3$ we can compare Monte Carlo simulation to the prices in (Johnson 1987); see for example Figure 1.

## 6.1 Maximum Payoffs

We will first price the derivative that has payoff $\max(S_1, S_2, \ldots, S_n)$, where the $S_i$ satisfy the usual properties. In fact, this is notationally quite cumbersome, and all the ideas are encapsulated in any reasonably small value of $n$, so we choose $n = 4$ (as we will see later, the fourth asset will be the strike).

Firstly, the value of the derivative is the sum of the value of 4 other derivatives, the $i^{th}$ of which pays $S_i(T)$ if $S_i(T) > S_j(T)$ for $j \neq i$, and 0 otherwise. Let us value the first of these, the others will have similar values just by cycling the coefficients.

We are considering the asset that pays $S_1(T)$ if $S_1(T)$ is the largest price. Now let $S_1$ be the numeraire asset with associated martingale measure $\mathbb{Q}_1$. We see that the value of the derivative is

$$
\begin{aligned}
V_1(t) &= S_1(t)e^{-q_1\tau}\mathbb{E}_t^{\mathbb{Q}_1}[1; S_{2/1}(T) < 1,\ S_{3/1}(T) < 1,\ S_{4/1}(T) < 1] \\
&= S_1(t)e^{-q_1\tau}\mathbb{Q}_1[S_{2/1}(T) < 1,\ S_{3/1}(T) < 1,\ S_{4/1}(T) < 1] \\
&= S_1(t)e^{-q_1\tau}\mathbb{Q}_1[\ln S_{2/1}(T) < 0,\ \ln S_{3/1}(T) < 0,\ \ln S_{4/1}(T) < 0]
\end{aligned} \tag{6}
$$

where $S_{i/j}(T) = \frac{S_i(T)}{S_j(T)}$.

Let $\sigma_{i/j} = \|\underline{a}_i - \underline{a}_j\|$. We know that under $\mathbb{Q}_j$ we have $\frac{dS_{i/j}}{S_{i/j}} = (q_j - q_i)dt + (\underline{a}_i - \underline{a}_j) \cdot dW^j$, so $\ln S_{i/j}(T) \sim \phi(\ln S_{i/j}(t) + (q_j - q_i - \frac{1}{2}\sigma_{i/j}^2)\tau, \sigma_{i/j}\sqrt{\tau})$.

Note that, and define

$$
\begin{aligned}
\sigma_{i/j}^2 &= \sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j \\
d_\pm^{i/j} &= \frac{\ln\dfrac{S_i(t)}{S_j(t)} + \left(q_j - q_i \pm \dfrac{1}{2}\sigma_{i/j}^2\right)(\tau)}{\sigma_{i/j}\sqrt{\tau}} \\
d_\pm^i &= \frac{\ln\dfrac{S_i(t)}{K} + \left(r - q_i \pm \dfrac{1}{2}\sigma_i^2\right)(\tau)}{\sigma_i\sqrt{\tau}}
\end{aligned}
$$

Hence $\mathbb{Q}_j[S_{i/j}(T) <> 1] = N(\mp d_-^{i/j})$.

Note that $d_\pm^{i/j} = -d_\mp^{j/i}$.

Also, the correlation between $S_{i/k}(T)$ and $S_{j/k}(T)$ is

$$
\begin{aligned}
\rho_{ij,k} &:= \frac{(\underline{a}_i - \underline{a}_k) \cdot (\underline{a}_j - \underline{a}_k)}{\|\underline{a}_i - \underline{a}_k\|\|\underline{a}_j - \underline{a}_k\|} \\
&= \frac{\underline{a}_i \cdot \underline{a}_j - \underline{a}_i \cdot \underline{a}_k - \underline{a}_k \cdot \underline{a}_j + \sigma_k^2}{\sqrt{(\sigma_i^2 + \sigma_k^2 - 2\underline{a}_i \cdot \underline{a}_k)(\sigma_j^2 + \sigma_k^2 - 2\underline{a}_j \cdot \underline{a}_k)}} \\
&= \frac{\rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k - \rho_{kj}\sigma_k\sigma_j + \sigma_k^2}{\sqrt{(\sigma_i^2 + \sigma_k^2 - 2\rho_{ik}\sigma_i\sigma_k)(\sigma_j^2 + \sigma_k^2 - 2\rho_{jk}\sigma_j\sigma_k)}}
\end{aligned} \tag{7}
$$

Hence $\mathbb{Q}_1[\ln S_{2/1}(T) < 0,\ \ln S_{3/1}(T) < 0,\ \ln S_{4/1}(T) < 0] = N_3(-d_-^{2/1}, -d_-^{3/1}$,-$d_-^{4/1}, \Omega_1)$ where $\Omega_1$, $\Omega_2$, $\Omega_3$ and $\Omega_4$ are $3 \times 3$ matrices; the simplest way to think of them is that they are initially $4 \times 4$ matrices, with $\Omega_k$ having $\rho_{ij,k}$ in the $(i, j)^{th}$ position, and then the $k^{th}$ row and $k^{th}$ column are removed.

Thus, the value of the derivative that pays off the largest asset is

$$
\begin{aligned}
V_{\max}(t) = {} & S_1(t)e^{-q_1\tau}N_3(-d_-^{2/1}, -d_-^{3/1}, -d_-^{4/1}, \Omega_1) \\
& + S_2(t)e^{-q_2\tau}N_3(-d_-^{1/2}, -d_-^{3/2}, -d_-^{4/2}, \Omega_2) \\
& + S_3(t)e^{-q_3\tau}N_3(-d_-^{1/3}, -d_-^{2/3}, -d_-^{4/3}, \Omega_3) \\
& + S_4(t)e^{-q_4\tau}N_3(-d_-^{1/4}, -d_-^{2/4}, -d_-^{3/4}, \Omega_4) \\
= {} & S_1(t)e^{-q_1\tau}N_3(-d_-^{2/1}, -d_-^{3/1}, -d_-^{4/1}, \rho_{23,1}, \rho_{24,1}, \rho_{34,1}) \\
& + S_2(t)e^{-q_2\tau}N_3(-d_-^{1/2}, -d_-^{3/2}, -d_-^{4/2}, \rho_{13,2}, \rho_{14,2}, \rho_{34,2}) \\
& + S_3(t)e^{-q_3\tau}N_3(-d_-^{1/3}, -d_-^{2/3}, -d_-^{4/3}, \rho_{12,3}, \rho_{14,3}, \rho_{24,3}) \\
& + S_4(t)e^{-q_4\tau}N_3(-d_-^{1/4}, -d_-^{2/4}, -d_-^{3/4}, \rho_{12,4}, \rho_{13,4}, \rho_{23,4})
\end{aligned} \tag{8}
$$

## 6.2 Best and Worst of Call Options

Let us start with the case where the payoff is the best of assets or cash. The payoff at expiry is $\max(S_1,\ S_2,\ S_3,\ K)$. If we consider this to be the best of four assets, where the fourth asset satisfies $S_4(t) = Ke^{-r\tau}$ and has zero volatility, then we recover the value of this option from §6.1. This fourth asset not only has no volatility but also is independent of the other three assets.

Thus, $\underline{a}_4 = \underline{0}$, $\rho_{ij,4} = \rho_{ij}$, $\sigma_{i/4} = \sigma_i = \sigma_{4/i}$, $d_\pm^{i/4} = d_\pm^i$, $d_\pm^{4/i} = -d_\mp^i$. Thus

$$
\begin{aligned}
V_{\max}(t) = {} & S_1(t)e^{-q_1\tau}N_3(-d_-^{2/1}, -d_-^{3/1}, d_+^1, \rho_{23,1}, \rho_{24,1}, \rho_{34,1}) \\
& + S_2(t)e^{-q_2\tau}N_3(-d_-^{1/2}, -d_-^{3/2}, d_+^2, \rho_{13,2}, \rho_{14,2}, \rho_{34,2}) \\
& + S_3(t)e^{-q_3\tau}N_3(-d_-^{1/3}, -d_-^{2/3}, d_+^3, \rho_{12,3}, \rho_{14,3}, \rho_{24,3}) \\
& + Ke^{-r\tau}N_3(-d_-^1, -d_-^2, -d_-^3, \rho_{12}, \rho_{13}, \rho_{23})
\end{aligned} \tag{9}
$$

Now let us consider the rainbow call on the max option.

Recall, this has payoff $\max(\max(S_1, S_2, S_3) - K, 0)$. Note that

$$
\begin{aligned}
\max(\max(S_1, S_2, S_3) - K, 0) &= \max(\max(S_1, S_2, S_3), K) - K \\
&= \max(S_1, S_2, S_3, K) - K
\end{aligned}
$$

and so

$$
\begin{aligned}
V_{\mathrm{cmax}}(t) = {} & S_1(t)e^{-q_1\tau}N_3(-d_-^{2/1}, -d_-^{3/1}, d_+^1, \rho_{23,1}, \rho_{24,1}, \rho_{34,1}) \\
& + S_2(t)e^{-q_2\tau}N_3(-d_-^{1/2}, -d_-^{3/2}, d_+^2, \rho_{13,2}, \rho_{14,2}, \rho_{34,2}) \\
& + S_3(t)e^{-q_3\tau}N_3(-d_-^{1/3}, -d_-^{2/3}, d_+^3, \rho_{12,3}, \rho_{14,3}, \rho_{24,3}) \\
& - Ke^{-r\tau}[1 - N_3(-d_-^1, -d_-^2, -d_-^3, \rho_{12}, \rho_{13}, \rho_{23})]
\end{aligned} \tag{10}
$$

Finally, we have the rainbow call on the min option. (Recall, this has payoff $\max(\min(S_1, S_2, S_3) - K, 0)$.) Because of the presence of both a maximum and minimum function, new ideas are needed. As before we first value the derivative whose payoff is $\max(\min(S_1, S_2, S_3), S_4)$.

If $S_4$ is the worst performing asset, then the payoff is the second worst performing asset. For $1 \le i \le 3$ the value of this payoff can be found by using asset $S_i$ as the numeraire. For example, the value of the derivative that pays $S_1$, if $S_4$ is the worst and $S_1$ the second worst performing asset, is

$$
S_1(t)e^{-q_1\tau}N_3(d_-^{2/1}, d_-^{3/1}, -d_-^{4/1}, \rho_{23,1}, -\rho_{24,1}, -\rho_{34,1})
$$

If $S_4$ is not the worst performing asset, then the payoff is $S_4$. Now the probability that $S_4$ is the worst performing asset is

$$
N_3(d_-^{1/4}, d_-^{2/4}, d_-^{3/4}, \rho_{12,4}, \rho_{13,4}, \rho_{23,4})
$$

and so the value of the derivative that pays $S_4$, if $S_4$ is not the worst performing asset, is

$$
S_4(t)e^{-q_4\tau}[1 - N_3(d_-^{1/4}, d_-^{2/4}, d_-^{3/4}, \rho_{12,4}, \rho_{13,4}, \rho_{23,4})]
$$

Thus, the value of the derivative whose payoff is $\max(\min(S_1, S_2, S_3), S_4)$ is

$$
\begin{aligned}
V(t) = {} & S_1(t)e^{-q_1\tau}N_3(d_-^{2/1}, d_-^{3/1}, -d_-^{4/1}, \rho_{23,1}, -\rho_{24,1}, -\rho_{34,1}) \\
& + S_2(t)e^{-q_2\tau}N_3(d_-^{1/2}, d_-^{3/2}, -d_-^{4/2}, \rho_{13,2}, -\rho_{14,2}, -\rho_{34,2}) \\
& + S_3(t)e^{-q_3\tau}N_3(d_-^{1/3}, d_-^{2/3}, -d_-^{4/3}, \rho_{12,3}, -\rho_{14,3}, -\rho_{24,3}) \\
& + S_4(t)e^{-q_4\tau}[1 - N_3(d_-^{1/4}, d_-^{2/4}, d_-^{3/4}, \rho_{12,4}, \rho_{13,4}, \rho_{23,4})]
\end{aligned} \tag{11}
$$

Hence the derivative with payoff $\max(\min(S_1, S_2, S_3), K)$ has value

$$
\begin{aligned}
V(t) = {} & S_1(t)e^{-q_1\tau}N_3(d_-^{2/1}, d_-^{3/1}, d_+^1, \rho_{23,1}, -\rho_{24,1}, -\rho_{34,1}) \\
& + S_2(t)e^{-q_2\tau}N_3(d_-^{1/2}, d_-^{3/2}, d_+^2, \rho_{13,2}, -\rho_{14,2}, -\rho_{34,2}) \\
& + S_3(t)e^{-q_3\tau}N_3(d_-^{1/3}, d_-^{2/3}, d_+^3, \rho_{12,3}, -\rho_{14,3}, -\rho_{24,3}) \\
& + Ke^{-r\tau}[1 - N_3(d_-^1, d_-^2, d_-^3, \rho_{12}, \rho_{13}, \rho_{23})]
\end{aligned} \tag{12}
$$

and the call on the minimum has value

$$
\begin{aligned}
V_{\mathrm{cmin}}(t) = {} & S_1(t)e^{-q_1\tau}N_3(d_-^{2/1}, d_-^{3/1}, d_+^1, \rho_{23,1}, -\rho_{24,1}, -\rho_{34,1}) \\
& + S_2(t)e^{-q_2\tau}N_3(d_-^{1/2}, d_-^{3/2}, d_+^2, \rho_{13,2}, -\rho_{14,2}, -\rho_{34,2}) \\
& + S_3(t)e^{-q_3\tau}N_3(d_-^{1/3}, d_-^{2/3}, d_+^3, \rho_{12,3}, -\rho_{14,3}, -\rho_{24,3}) \\
& - Ke^{-r\tau}N_3(d_-^1, d_-^2, d_-^3, \rho_{12}, \rho_{13}, \rho_{23})
\end{aligned} \tag{13}
$$

## 7. Finding the Value of Puts

This is easy, because put-call parity takes on a particularly useful role. It is always the case that

$$
V_{\mathrm{c}}(K) + Ke^{-r\tau} = V_{\mathrm{p}}(K) + V_c(0) \tag{14}
$$

where the parentheses denotes strike. $V$ could be an option on the minimum, the maximum, or indeed any ordinal of the basket. If we have a formula for $V_{\mathrm{c}}(K)$, as established in one of the previous sections, then we can evaluate $V_{\mathrm{c}}(0)$ by taking a limit as $K \downarrow 0$, either formally (using facts of the manner $N_2(x, \infty, \rho) = N_1(x)$ and $N_3(x, y, \infty, \Sigma) = N_2(x, y, \rho_{xy})$) or informally (by forcing our code to execute with a value of K which is very close to, but not equal to, 0 - thus avoiding division by 0 problems but implicitly implementing the above-mentioned fact). By rearranging, we have the put value.

## 8. Deltas of Rainbow Options

By inspecting (9) one might expect that

$$
\frac{\partial V_{\max}}{\partial S_1} = e^{-q_1\tau}N_3(-d_-^{2/1}, -d_-^{3/1}, d_+^1, \rho_{23,1}, \rho_{24,1}, \rho_{34,1})
$$

with similar results holding for $\frac{\partial V_{\max}}{\partial S_2}$ and $\frac{\partial V_{\max}}{\partial S_3}$, and indeed for the dual delta $\frac{\partial V_{\max}}{\partial K}$.

Thus turns out to be true in this case, but to claim it as an 'obvious fact' would be erroneous. Recall Euler's Homogeneous Function Theorem, which we will cast in our case of a function of four variables $V(x_1, x_2, x_3, x_4)$. The theorem states that if $V(\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4) = \lambda V(x_1, x_2, x_3, x_4)$ for any constant $\lambda$ then $V(x_1, x_2, x_3, x_4) = x_1 \frac{\partial V}{\partial x_1} + x_2 \frac{\partial V}{\partial x_2} + x_3 \frac{\partial V}{\partial x_3} + x_4 \frac{\partial V}{\partial x_4}$.

The argument of (Johnson 1987) is essentially an application of this theorem: he intuits what $\frac{\partial V}{\partial S_i}$ is and then 'reassembles' $V$ using this result.

However, to claim a converse of the form that if $V(x_1, x_2, x_3, x_4) = x_1 \Omega_1 + x_2\Omega_2 + x_3\Omega_3 + x_4\Omega_4$ for some 'nice' $\Omega_i$ then of necessity $\Omega_i = \frac{\partial V}{\partial x_i}$ is false. (6) provides a counterexample; because certainly it is not the case that $\frac{\partial V}{\partial S_i} = 0$ for $i > 1$. To jump at the claim that it is an obvious fact that the above is the formula for delta is probably an application of this false converse.

However, these claims are true in the case of $V_{\max}$, as it is for $V_{\mathrm{cmax}}$, $V_{\mathrm{cmin}}$, $V_{\mathrm{pmax}}$ and $V_{\mathrm{pmin}}$.

## 9 Finding the Capital Guarantee on the 'Best of Assets or Cash Option'

We wish to determine the strike $K$ of the 'best of assets or cash' option so that at inception the valuation of the option is equal to $K$. Denoting the value of such an option as $V(K)$—implicitly fixing all other variables besides the strike—we wish to solve $V(K) = K$.

To do so using Newton's method is fortunately quite manageable, for the same reasoning that we have already seen. As previously promised we have from (9) that

$$\frac{\partial V}{\partial K} = e^{-r\tau} N_3(-d_-^1, -d_-^2, -d_-^3, \rho_{12}, \rho_{13}, \rho_{23})$$

Hence the appropriate Newton method iteration is

$$K_{n+1} = K_n - \frac{V(K_n) - K_n}{\frac{\partial V}{\partial K}|_{K=K_n} - 1} \tag{15}$$

and this is iterated to some desired level of accuracy. An alternative would be to iterate $K_{n+1} = V(K_n)$, our differentiation shows that the fund $V$ is a contraction, and so this iteration will converge to the fixed point $V(K) = K$ by the contraction mapping theorem.

It is important to note that the process of finding the fair theoretical strike is not just a curiosity. In the first place, it is attractive for the buyer of the option that they will get at least their premium back. (There is a floor on the return of 0%.) Moreover, if $K$ is this fair strike, the trader will strike the option at an $K^*$, where $K^* > K$, in order to expect fat in the deal.

To see this, we can construct in a complete market a simple arbitrage strategy: imagine that the dealer sells the client for $K^*$ an option struck at $K^*$, and hedges this with the 'fair' dealer by paying $K$ for an option struck at $K$.[1] The difference $K^* - K$ is invested in a risk free account for the expiry date. Three cases then arise:

- If $\max(S_1, S_2, S_3) \leq K$ then we owe $K^*$. The fair trader pays K and we obtain $K^* - K$ from saving, and profit from the time value of $K^* - K$.
- If $K < \max(S_1, S_2, S_3) \leq K^*$, then the fair trader pays $S_1$ say. We sell this, and obtain the balance to $K^*$ from saving.
- If $K^* < \max(S_1, S_2, S_3)$, then the fair trader pays $S_1$ say and we deliver this.

## 10 Pricing Rainbow Options in Reality

The model that has been developed here lies within the classical Black-Scholes framework. As is well known, the assumptions of that framework do not hold in reality; various stylised facts argue against that model. For vanilla options, the model is adjusted by means of the skew—this skew exactly ensures that the price of the option in the market is exactly captured by the model. Models which extract information from that skew and of how that skew will evolve are of paramount importance in modern mathematical finance.

After a moment's thought one will realise what a difficult task we are faced with in applying these skews here. Let us start by being completely naïve: we wish to mark our rainbow option to market by using the skews of the various underlyings. Firstly, what strike do we use for the underlying? How does the strike of the rainbow translate into an appropriate strike for an option on a single underlying? Secondly, suppose we somehow resolved this problem, and for a traded option, wished to know its implied volatility? A familiar problem arises: often the option will have two, sometimes even three different volatilities of one of the assets which recover the price (all other inputs being fixed). To be more mathematical, the map from volatility to price is not injective, so the concept of implied volatility is ill defined. See Figure 2.

To see the sensitivity to the inputs, suppose to the setup in Figure 2 we add a third asset as elaborated in Figure 3. Of course the general level

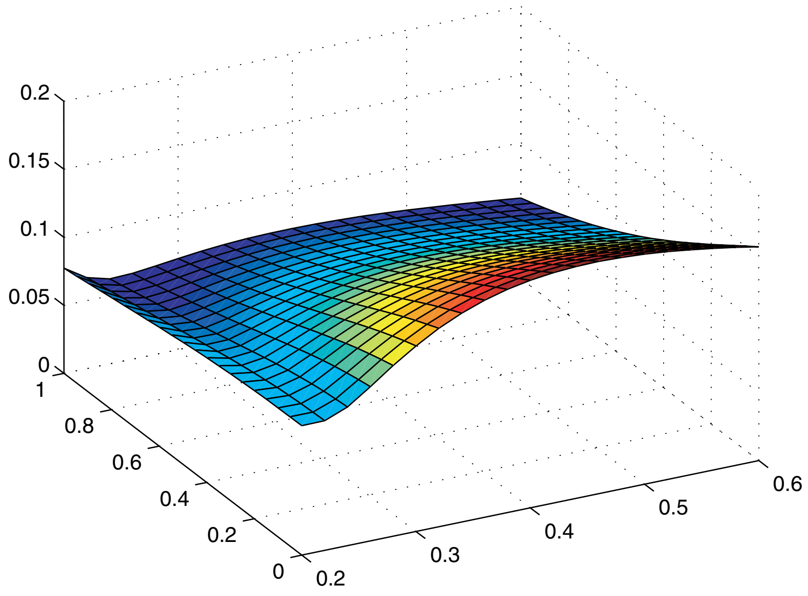


**Figure 2: The price for a call on the minimum of two assets. $S_1 = 2$, $S_2 = 1$, $K = 1$, $\tau = 1$, $r = 10\%$, $\rho = -70\%$, $20\% \leq \sigma_1 \leq 60\%$, $\sigma_2 \leq 100\%$.**

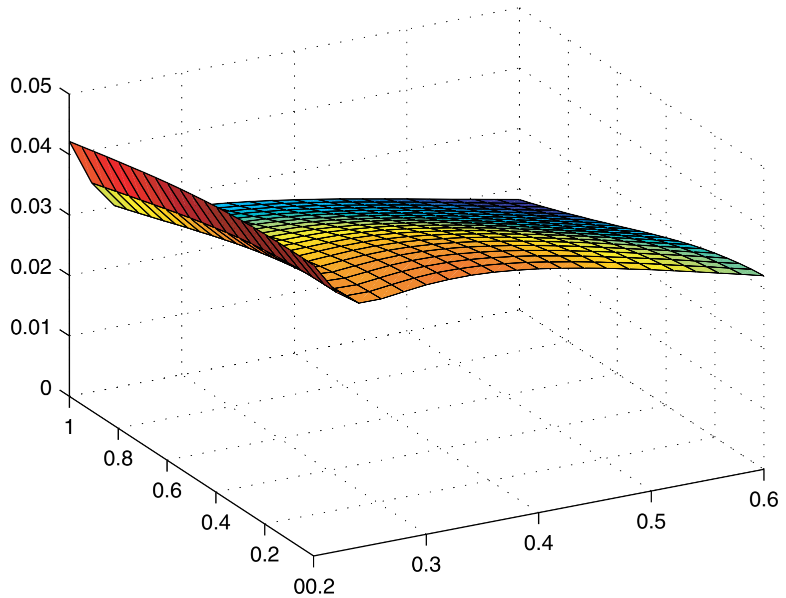


**Figure 3: The price for a call on the minimum of three assets. As above, in addition $S_3 = 1$, $S_2 = 1$, $K = 1$, $\tau = 1$, $r = 10\%$, $20\% \leq \sigma_1 \leq 60\%$, $\sigma_2 \leq 100\%$ $\sigma_3 \leq 30\%$ fixed, correlation structure $\rho_{12} = -70\%$, $\rho_{13} = 30\%$, $\rho_{23} = -20\%$.**

of the value of the asset changes, but so does the entire geometry of the price surface.

Another issue is that of the assumed correlation structure: again, correlation is difficult to measure; if there is implied data, then it will have a strike attached. Finally, the joint normality hypothesis of returns of prices will typically be rejected.

A popular approach is to use skews from the vanilla market to infer the marginal distribution of returns for each of the individual assets and then 'glue them together' by means of a copula function. Given a multivariate distribution of returns, rainbow options can then be priced by Monte Carlo methods.

## FOOTNOTES & REFERENCES

**1.** The 'fair' dealer is the perfect hedger, whose replicating portfolio ends up with exactly the payoff.

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