Empirical and Fitted Estimation of Risk Measures for Hedge Funds

Honours project in the Advanced Mathematics of Finance Programme.

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December 6, 2007

I acknowledge the help and advice, with thanks, of Barry du Toit of Peregrine Securities, Olav Jordans and Thomas McWalter of Wits University, Peter Urbani of Infiniti Capital, Joel Heinrich of the University of Pennsylvania and Graeme West of Financial Modelling Agency.

Abstract

In this project we consider some risk-adjusted performance measures for hedge fund returns. In Chapter 2 we briefly discuss the well-known risk-adjusted measures Sharpe and Sortino, along with a relatively new risk-adjusted measure Omega, which is looked at in more detail. We also consider the Value-at-Risk and Expected Shortfall risk measures. A general method for optimising a risk measure is shown in Chapter 3. A discussion on the problems encountered when this method is implemented is included. It would be desirable to calculate the risk measures introduced in Chapter 2 under some distribution assumption. Recently hedge funds have shown some interest in the Pearson Type IV distribution. Therefore in Chapter 4, an in-depth discussion of the mathematics of the Pearson Type IV distribution is given. Much of what appears here is original mathematics. Various issues regarding the implementation of these calculations are discussed. Chapter 5 revisits the measures introduced in Chapter 2 applied to the Pearson Type IV distribution. Finally, in Chapter 6, the results of the implementation of the empirical and Pearson Type IV are compared using South African and international hedge fund data.

Contents

Chapter 1

Introduction

A naïve approach to comparing financial instruments is to compare their returns. However it is well-known that one should also compare the risk characteristics of the instruments as well. Thus one would like to consider the performance of the instruments but adjust our preferences according to the risk taken on by each one. In this project we consider the risk-adjusted performance measures Sharpe, Sortino and Omega as well as the risk measures Value-at-Risk and Expected Shortfall.

The risk-adjusted measures can be calculated empirically, but Value-at-Risk and Expected Shortfall cannot. Also, hedge fund data is sparse. Thus, it is necessary to fit the data to a distribution. The Pearson Type IV distribution is able to fit data with high peaks and fat tails as well as skewness and so is a good choice to fit hedge fund data; this distribution is discussed in detail in this project. We show how to calculate the risk-adjusted measures and Value-at-Risk and Expected Shortfall under the Pearson Type IV distribution.

The risk-adjusted measures are maximised by finding the optimal weights through time and a comparison between the empirical approach and the Pearson Type IV distributional assumption is given. Also, results on Value-at-Risk and Expected Shortfall calculations done under the Pearson Type IV distribution are shown.

Chapter 2

Risk Measures

In this chapter, a selection of the most common risk measures applied in the hedge fund industry is discussed. The risk-adjusted performance measures Sharpe, Sortino and Omega are looked at first. Then a mechanism for calculating the risk measures Value-at-Risk and Expected Shortfall, which are standard tools in the finance industry, are introduced in the context of hedge funds.

2.1 Sharpe

One of the most popular risk-adjusted measures is the Sharpe ratio. The Sharpe ratio, developed by Sharpe (1966), is given by

$$\text{Sharpe} = \frac{\mu - r}{\sigma} \tag{2.1}$$

where the risk-free return is r, the mean of the distribution of returns is μ and the standard deviation of returns is σ. Since the Sharpe ratio is time-dependent, care needs to be taken to ensure that r, μ and σ are expressed on the same basis (in (2.1) r, μ and σ are expressed annually). The higher the Sharpe ratio, the higher the return for the risk that was taken. Thus a higher ratio is better. For an in-depth discussion on the Sharpe ratio see (Lhabitant, 2004, §4.1).

2.2 Sortino

The Sortino ratio, devised by Frank Sortino (F. Sortino and Platinga (1999)), is given by

$$\text{Sortino } = \frac{\mu - r_{\text{MA}}}{\sigma_{\text{D}}} \tag{2.2}$$

where r_{MA} is the minimum acceptable return, μ is the mean of the distribution of returns and σ_{D} is the standard deviation of the returns below the minimum acceptable return, also called the *downside deviation.* In (2.2) r_{MA}, μ and σ_{D} are expressed annually. This risk-adjusted measure does not measure risk in terms of the volatility of the returns but rather in terms of the downside deviation. The choice of r_{MA} depends on how risk-averse the investor is: a higher r_{MA} corresponds to a higher risk-aversion. Since different choices of r_{MA} produces very different results, r_{MA} should always be reported along with the Sortino ratio. Also, when comparing funds, one should ensure that the same r_{MA} is used. Again, more detail on the Sortino ratio can be obtained from (Lhabitant, 2004, §4.4.1).

2.3 Omega

The Omega ratio is a fairly recent risk-adjusted measure introduced by Keating and Shadwick (April 2002). It is the ratio of probability weighted gains to probability weighted losses of a returns distribution set at some chosen threshold level. When this threshold level is varied, the Omega function is obtained. It is defined by

$$\Omega(L) = \frac{\int_L^b 1 - F(x)dx}{\int_a^L F(x)dx} \tag{2.3}$$

where $F(x)$ is the cumulative density function of the returns distribution, (a, b) the interval of returns (this interval can be $(-\infty, \infty)$) and L the chosen threshold level.
In Figure 2.1 the cumulative distribution function of a series of returns is shown. Here $\int_L^b 1 - F(x)dx$ represents the probability weighted gains and $\int_a^L F(x)dx$ the probability weighted losses given threshold level L. Intuitively, the Omega function can be seen as a bet - the sum of the potential winnings multiplied by their corresponding probabilities is divided by the sum of the potential losses multiplied by their corresponding losses. The level at which we differentiate between winning and losing, the threshold level L, depends on the individual: again it corresponds to risk aversion.

2.3.1 Discrete Calculations

The cumulative distribution function can be constructed empirically from historical returns. According to Keating and Shadwick (April 2002), "Another important aspect is that [the Omega function] is not plagued by sampling uncertainty, unlike standard statistical estimators - as it is calculated directly from the observed distribution and requires no estimates." This is quite a disingenuous statement. What it fails to take into account is that the observed distribution is in

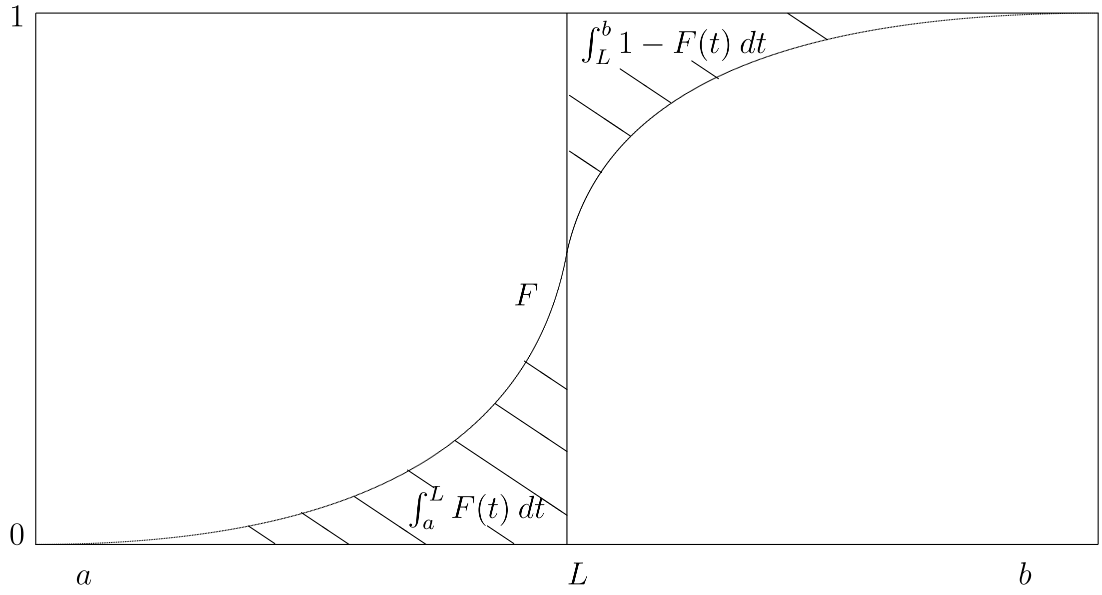

Figure 2.1: The calculation of Omega.

itself observations from the actual unknown distribution. Thus the Omega function calculated is in itself just another sample statistic.

Suppose we have n returns and m is the maximum index of the return less or equal to L. In Figure 2.2 the empirical cumulative distribution function of a returns distribution is shown where $n = 10$ and $m = 7$.

There are several ways in which the Omega ratio can be calculated. A first approach would be to calculate the Riemann sums as seen in Figure 2.3:

$$\int_a^L F(x)dx = \sum_{i=1}^{m-1}(x_{i+1} - x_i)i + (L - x_m)m$$

and

$$\int_L^b 1 - F(x)dx = \sum_{i=m+1}^{n-1}(x_{i+1} - x_i)(n - i) + (x_{m+1} - L)(n - m).$$

The above calculations can be simplified by calculating the 'horizontal Riemann sums' instead as shown in Figure 2.4.

$$\int_a^L F(x)dx = \sum_{i=1}^{m}(L - x_i) = mL - \sum_{i=1}^{m} x_i$$

and

$$\int_L^b 1 - F(x)dx = \sum_{i=m+1}^{n}(x_i - L) = \sum_{i=m+1}^{n} x_i - (n - m)L.$$

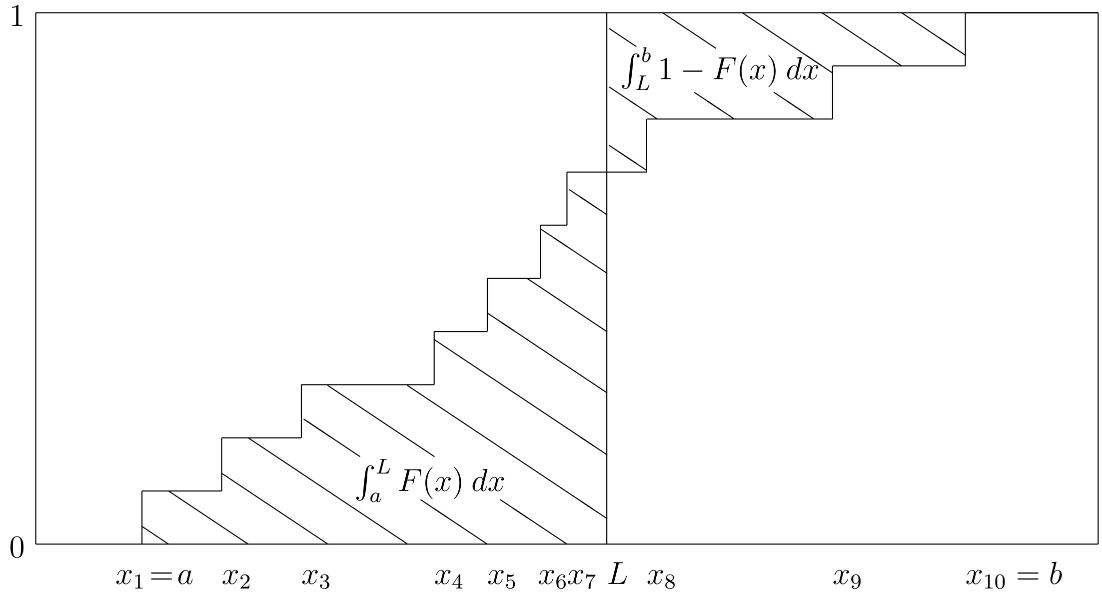

Figure 2.2: Empirical cumulative distribution function with the required areas of integration indicated.

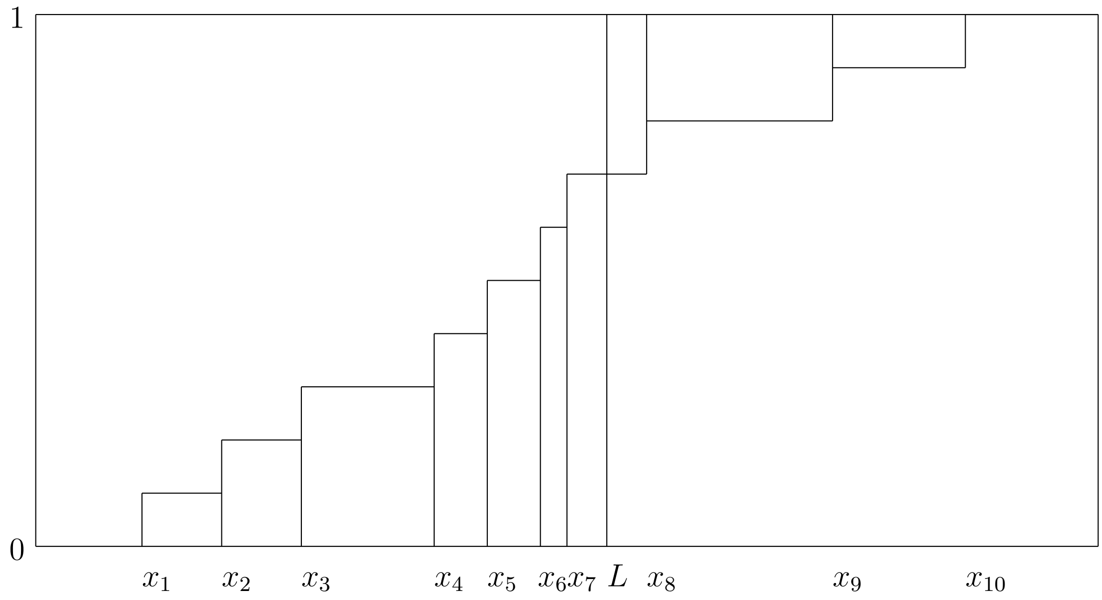

Figure 2.3: Calculation of the Omega function using Riemann sums. Here, $n = 10$, $m = 7$.

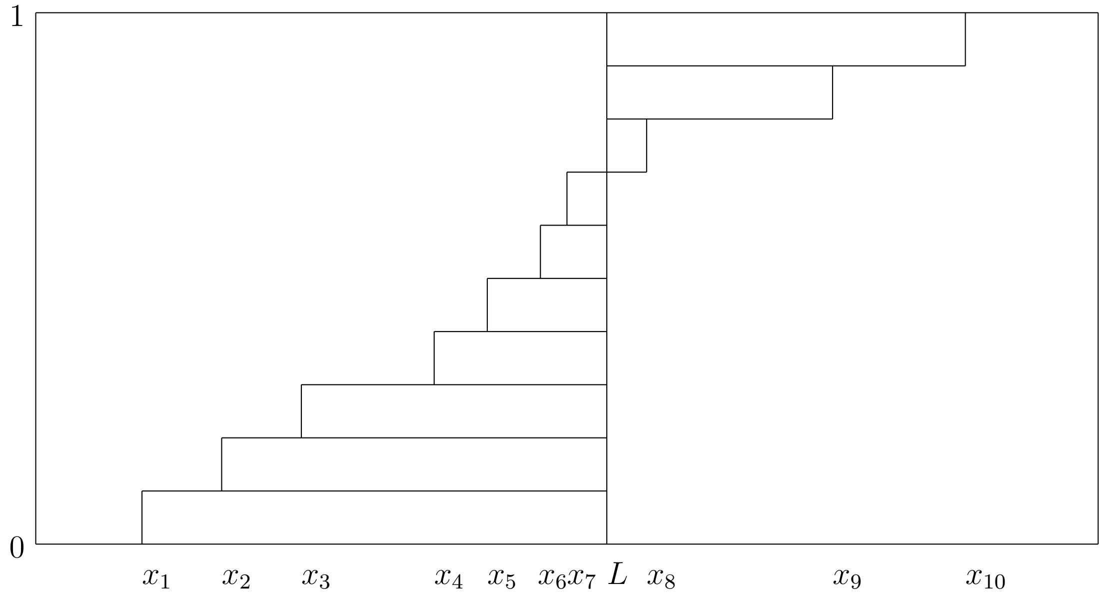

Figure 2.4: Calculation of the Omega function using 'horizontal Riemann sums' rather than vertical ones. Here, $n = 10$, $m = 7$.

When implementing the Omega function in a programming language, care should be taken when the threshold level is less than the smallest return in the portfolio. Omega should then be equal to infinity. Infinity 'as a number' is not always supported by the programming language, so it would have to be set to some very high value. Another important fact that should be remembered is that the portfolio of returns should be sorted in ascending order.

2.3.2 Mathematical Properties

Clearly $\Omega(L)$ is non-negative and strictly decreasing. Furthermore we see that if L is set below the lowest return there are no losses, so that $\Omega(L)$ becomes infinity, and when L is set above the highest return we have no gains, and $\Omega(L)$ is zero. Another interesting property is given in Proposition 1.

Proposition 1. If the threshold level L is equal to the mean of the distribution, then $\Omega(L)$ is equal to one.

Proof. Suppose that the distribution has support contained in $[a, b]$. Let the mean of the distribution be given by μ, then

$$\begin{aligned}
\mu &= \int_a^b xf(x)dx \\
&= [xF(x)]_a^b - \int_a^b F(x)dx \\
&= b - \int_a^b F(x)dx
\end{aligned}$$

If $\Omega(L) = 1$, then

$$\begin{aligned}
\int_a^L F(x)dx &= \int_L^b (1 - F(x))dx \\
&= b - L - \int_L^b F(x)dx \\
\Rightarrow L &= b - \int_a^b F(x)dx
\end{aligned}$$

and we have that $L = \mu$.

When the support is infinite, a limiting argument can be used to derive the same result. □

Proposition 1 does not only hold when we know the probability distribution function but also when we only have a sample from an unknown distribution of returns. To see this, suppose the sample is $x_1, \ldots, x_n$. Consider the atomic probability measure μ where the atoms are $x_1, \ldots, x_n$ and each atom has measure $\frac{1}{n}$. Then in the above calculations simply replace the Lebesgue measure with the atomic measure.

As pointed out by du Toit (2005), if the Omega function of a fund A is always above that of another, fund B (we say that fund A *dominates* fund B), we would always prefer fund A to fund B. Thus given a threshold level, we would naturally prefer the fund with the higher Omega ratio at this level. Now consider the Omega functions in Figure 2.5 for two funds X and Y as shown in Keating and Shadwick (April 2002). Funds X and Y were chosen from the EDHEC Alternative Indexes provided by EDHEC-Risk (2007). From Proposition 1 we see that fund X has a higher mean return than fund Y. The Omega functions of the two funds cross at C. In fact, the Omega function of two funds may cross several times, so that our choice of fund might change for different threshold levels.

Note that fund X is riskier than fund Y. For low threshold levels we see that the $\Omega_{\mathrm{X}}(L)$ is less than $\Omega_{\mathrm{Y}}(L)$ which means that the denominator in the calculation of $\Omega_{\mathrm{X}}(L)$ is relatively big compared to that of $\Omega_{\mathrm{Y}}(L)$ indicating that the distribution of fund X has a fatter left tail than that of

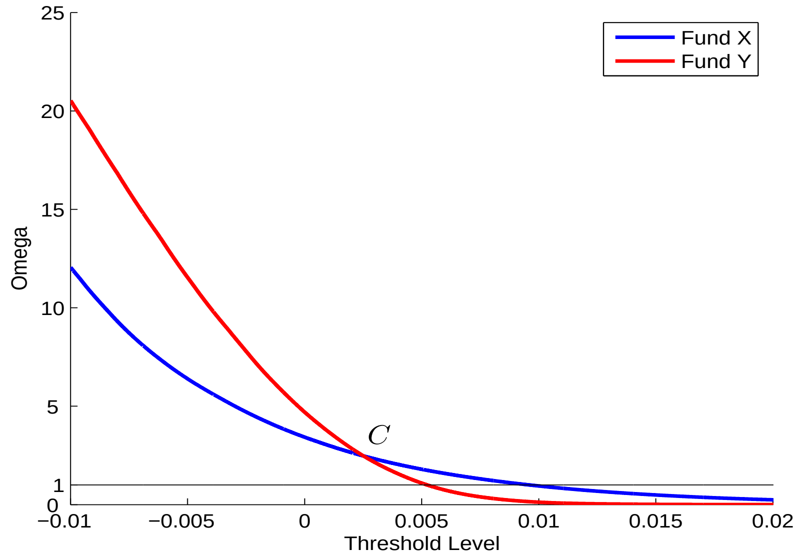

Figure 2.5: The Omega calculated for fund X and fund Y for different threshold levels.

fund Y. Thus if we choose a threshold level to the left of C, fund Y would be preferable. But a threshold level to the right of C would result in a choice of X.

Keating and Shadwick (April 2002) claim that all information on a distribution is encoded in the Omega function. In particular, they claim that the distribution function can be recovered from the Omega function, but they do not give a prescription for this. In Theorem 1 we solve this problem. But first we provide a lemma which will be of use in this theorem. This lemma uses ideas from the proof of (Cascon et al., 2003, Theorem 1(8)); it is not clear that they explicitly knew a result of this form.

Lemma 1. Suppose $F(\cdot)$ is a cumulative distribution function with mean μ, then

$$\int_L^\infty 1 - F(t)dt - \int_{-\infty}^L F(t)dt = \mu - L.$$

Proof. Let $N(F,L) = \int_L^\infty 1 - F(t)dt$ and $D(F,L) = \int_{-\infty}^L F(t)dt$. Then

$$\begin{aligned}
&N(F,L) - D(F,L) \\
=&N(F,L) - N(F,\mu) + D(F,\mu) - D(F,L) \\
=&\int_L^\infty 1 - F(t)dt - \int_\mu^\infty 1 - F(t)dt + \int_{-\infty}^\mu F(t)dt - \int_{-\infty}^L F(t)dt \\
=&\int_L^\mu 1 - F(t)dt + \int_L^\mu F(t)dt \\
=&\mu - L.
\end{aligned}$$

□

Theorem 1. The distribution function can be determined from the Omega function. This relationship is given by

$$f(L) = \frac{\partial^2}{\partial L^2}\left[\frac{\mu_F - L}{\Omega_F(L) - 1}\right]. \tag{2.4}$$

Proof.

$$\Omega_F(L) = \frac{N(F,L)}{D(F,L)} = \frac{\mu_F - L + D(F,L)}{D(F,L)} = \frac{\mu_F - L}{D(F,L)} + 1.$$

and so

$$D(F,L) = \frac{\mu_F - L}{\Omega_F(L) - 1}.$$

Hence, by the Fundamental Theorem of Calculus

$$f(L) = \frac{\partial^2}{\partial L^2}\left[\frac{\mu_F - L}{\Omega_F(L) - 1}\right].$$

□

2.4 Value-at-Risk and Expected Shortfall

Value-at-Risk and Expected Shortfall have not been used in the hedge fund industry. Because hedge fund data is sparse, it is impossible to calculate Value-at-Risk and Expected Shortfall empirically. Nevertheless we can compute these measures when a distributional assumption is made. In Chapter 4, where the Pearson Type IV distribution is discussed in detail, various results are derived which will allow us to calculate Value-at-Risk and Expected Shortfall under this distributional assumption (as well as the Sharpe and Sortino ratios, and Omega).

In order to calculate risk measures from hedge fund data the profit and loss numbers for the hedge funds may be required. Hedge fund data is provided in the form of monthly returns. This is not a problem when the returns provided are simple

$$\begin{aligned} r(t+1) &= \frac{V(t+1)-V(t)}{V(t)} \\ \Rightarrow r(t+1)V(t) &= V(t+1)-V(t). \end{aligned} \tag{2.5}$$

Here $r(t+1)$ indicates the return over time period $[t, t+1]$ and $V(t)$ the value of a fund at time t. Now $V(t+1) - V(t)$ is the profit and loss of a fund over the time period $[t, t+1]$.
Let X be the distribution for $r(t+1)$. Define x_α to be the α^{th} percentile of the returns distribution at that time t, that is, $\mathbb{P}[X \leq x_\alpha] = \alpha$. x_α is found as the cumulative inverse function evaluated at α; in the case that the distribution function is continuous, there is no possible ambiguity here as there would be in the case of a discrete or mixed distribution. Then the Value-at-Risk is given by

$$\text{VaR} = -V(t)x_\alpha.$$

Using Bayes' formula we calculate the Expected Shortfall as follows

$$\text{ES} = -V(t)\mathbb{E}[X|X \leq x_\alpha] = -V(t)\frac{\mathbb{E}[X\mathbb{1}_{\{X \leq x_\alpha\}}]}{\mathbb{P}[X \leq x_\alpha]} = -\frac{V(t)}{\alpha}\mathbb{E}[X\mathbb{1}_{\{X \leq x_\alpha\}}]. \tag{2.6}$$

Chapter 3

Optimisation

3.1 Optimising Risk-Adjusted Measures

Suppose we want to construct a portfolio P from a group of hedge funds $F_j,\ j = 1,\ 2,\ \ldots,\ m$. We would like to find the weights $\mathbf{w} = (w_1,\ \ldots,\ w_m)'$ in which each fund should be held in order to maximise the risk-adjusted measure. This optimisation is subject to the following contraints

$$\sum_{j=1}^{m} w_j = 1 \text{ and} \tag{3.1}$$

$$w_j \geq 0\ \forall j. \tag{3.2}$$

(3.2) is necessary when dealing with hedge funds because one cannot hold a short position in a hedge fund. The return vector of our portfolio is given by

$$\mathbf{R}^P = R^F \mathbf{w}.$$

Here R^F is the $n \times m$ matrix of returns with R^F_{ij} the i'th return of fund F_j.
Matlab provides a function, `fmincon()`, which attempts to find a minimum of a function with constraints.

```
fmincon(@Measure,StartingWeights,[],[],UnitVector,1,ZeroVector,[],[],Options)
```

An explanation of each argument follows:

- `Measure()` is the function which we want to maximise. Since `fmincon()` is a minimising function, the value returned by `@Measure` is the negated value of the measure.

- `fmincon` requires an initial guess. In this case it is given by `StartingWeights` a vector of length m.

- The next two parameters are reserved for linear inequalities. Since we do not have any in this optimisation problem we leave them blank (empty brackets `[]` in Matlab).

- Our first constraint, (3.1), is introduced in the next two positions. (3.1) is equivalent to saying that

  $$\mathbf{u}.\mathbf{w} = 1$$

  holds where $\mathbf{u} = (1, 1, \ldots, 1)$ a row vector of length m. Here $\mathbf{u}$ is given by `UnitVector`.

- The next two parameters passed to `fmincon()` are the lower and upper bounds of the weight vector $\mathbf{w}$. (3.2) provides the lower boundary

  $$\mathbf{w} \geq \mathbf{0}$$

  where $\mathbf{0} = (0, 0, \ldots, 0)'$ (a column vector of length m) is represented by `ZeroVector`.

- Since our optimisation problem has no upper bound we leave it empty. Alternatively, we could have a constraint that the portfolio does not contain any weights that are more than a particular value. For example we could require that there is no weight greater than 50%. Then the empty brackets `[]` would be replaced by `0.5.*UnitVector`.

- The second last argument is for nonlinear inequalities. A function must be provided which returns values $g(\mathbf{w})$ and $h(\mathbf{w})$ such that

  $$g(\mathbf{w}) \leq 0$$
  $$h(\mathbf{w}) = 0.$$

  In the above this parameter is left empty, but we can also pass it `@Constraint` where `Constraint` is a function which returns the values $g(\mathbf{w})$ and $h(\mathbf{w})$. This approach is taken in §5.4.1.

- The last parameter, `Options`, contains the optimisation options specified. These refer to the stepping size in the search algorithm, the convergence tolerance, etc.

3.2 Choosing the Portfolio

A kind of back-testing is performed in order to determine if the mentioned risk-adjusted measures have any predictive power. Given a history of returns, we divide the data in half. The first half is used to find the weights in which each fund should be held so that the particular risk-adjusted measure is optimised (as was shown in §3.1). Then, using the set of weights for each measure, we set up a portfolio corresponding to each. The second half of the data is used to determine how well the chosen portfolios have done.

3.3 Rebalancing

Instead of optimising the measures and choosing the portfolios once, we may want to rebalance the weights monthly. This means that we would start optimising the different risk-adjusted measures given half the history of returns and then, every month, optimise the measures again. This new optimisation will then be on the extended history of returns and the next set of returns for the month. We then choose a new portfolio based on the new optimisation. Our analysis broadly shows two facts - this rebalancing does not significantly enhance performance, and the portfolio weights may be quite unstable through time (these results are shown in Chapter 6). Therefore rebalancing is not contributing to a better performance because of transaction costs. Also, monthly rebalancing might not be possible. Since hedge funds are not traded instruments, we cannot sell our holding immediately. A notice would have to be given and there might be withdrawal penalties. Another problem we are faced with is the fact that hedge fund data is sometimes reported two or three months after their occurrence.

Chapter 4

Mathematics of the Pearson IV Distribution

Recently there has been some interest in the hedge fund industry in using the Pearson Type IV distribution to fit returns data. This is no doubt due to its ability to fit data with high peaks and heavy tails. Furthermore, there have been in the last ten years significant breakthroughs in the mathematics of this distribution, which we discuss in this chapter.

The Pearson distributions were developed in order to fit observed distributions from data which display heavy tails and skewness of a wide range. In Pearson (1895), Karl Pearson introduced the first four types, I to IV, of distributions. This was in addition to the normal distribution originally known as the type V distribution. Later on, in Pearson (1901) the type V distribution was redefined and the type VI distribution was identified. The special cases and subtypes, VII to XII of the Pearson system was introduced in Pearson (1916). In this chapter we consider mainly the Pearson Type IV distribution. Historically this has been a distribution which has been difficult to use. All calculations were necessarily only approximations with integrals found for example by quadrature. The normalisation constant of the Pearson Type IV distribution was only found as late as Nagahara (1999). He could only find the cumulative distribution function in some cases. The most general solution first appears in Heinrich (2004).

In this section we refer to Heinrich (2004), Yan (2005) and Nagahara (1999). We have adopted the notation as used in Heinrich (2004). Symbol translation is given in Table 4.1.

Heinrich	Yan	Nagahara
λ	ξ	μ
a	λ	τ
m	m	b
ν	ν	$-2b\delta$

Table 4.1: Symbol translation.

4.1 Probability Density Function

The Pearson Type IV probability density function is given by

$$f(m,\nu,a,\lambda;x) = k(m,\nu,a)\left[1+\left(\frac{x-\lambda}{a}\right)^2\right]^{-m} e^{-\nu \arctan \frac{x-\lambda}{a}} \tag{4.1}$$

where $m > \frac{1}{2}$, ν, $a > 0$ and λ are real-valued parameters and $x \in \mathbb{R}$ [1]. Here $k(\cdot,\cdot,\cdot)$ is a normalisation constant chosen to ensure that this function is indeed a probability density function, that is, $\int_{-\infty}^{\infty} f(m,\nu,a,\lambda;x)dx = 1$.

The Pearson Type IV probability density function is shown in Figures 4.1 and 4.2 where one parameter is varied and the others are fixed. We now state some observations made from the graphs:

- As m increases, the probability density function becomes more peaked and as m gets closer to a half, the probability density function flattens very quickly. Thus m can be seen as a kurtosis parameter: with lower m one has greater kurtosis.
- When ν is varied we observe that the mean of the distribution moves. Also, there is positive skewness when $\nu > 0$, negative skewness when $\nu < 0$ and we have a symmetric probability density function when $\nu = 0$.
- We see that a is a scaling parameter.
- It is clear that λ is purely a location parameter.

[1] The special case of the cumulative distribution function that Nagahara (1999) solves is where $m = n$ and $m = n + \frac{1}{2}$, $n \in \mathbb{N}$ (see (Nagahara, 1999, §4)).

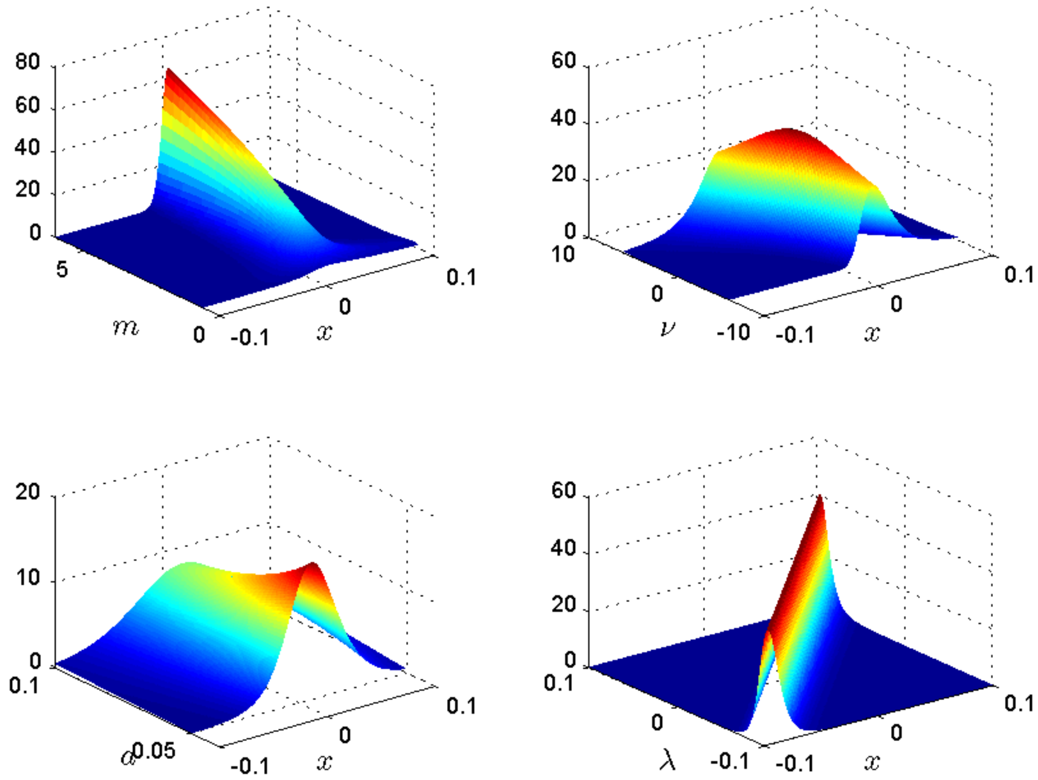

Figure 4.1: Surface plots of the Pearson Type IV probability density function in which one parameter is varied and the remaining three are fixed. In these surface plots the fixed parameters have values $m = 2.8$, $\nu = -0.8$, $a = 0.02$ and $\lambda = 0.004$.

4.2 Cumulative Distribution Function

In this section we follow the programme of Heinrich (2004) and determine the closed form formula for the cumulative distribution function. We first prove a lemma which will be useful in a later proposition.

Lemma 2. For $\phi \in \mathbb{R}$,

$$1 - e^{2i\phi} = -2ie^{i\phi} \sin \phi.$$

Proof. From Euler's formula

$$e^{2i\phi} = \cos 2\phi + i \sin 2\phi$$

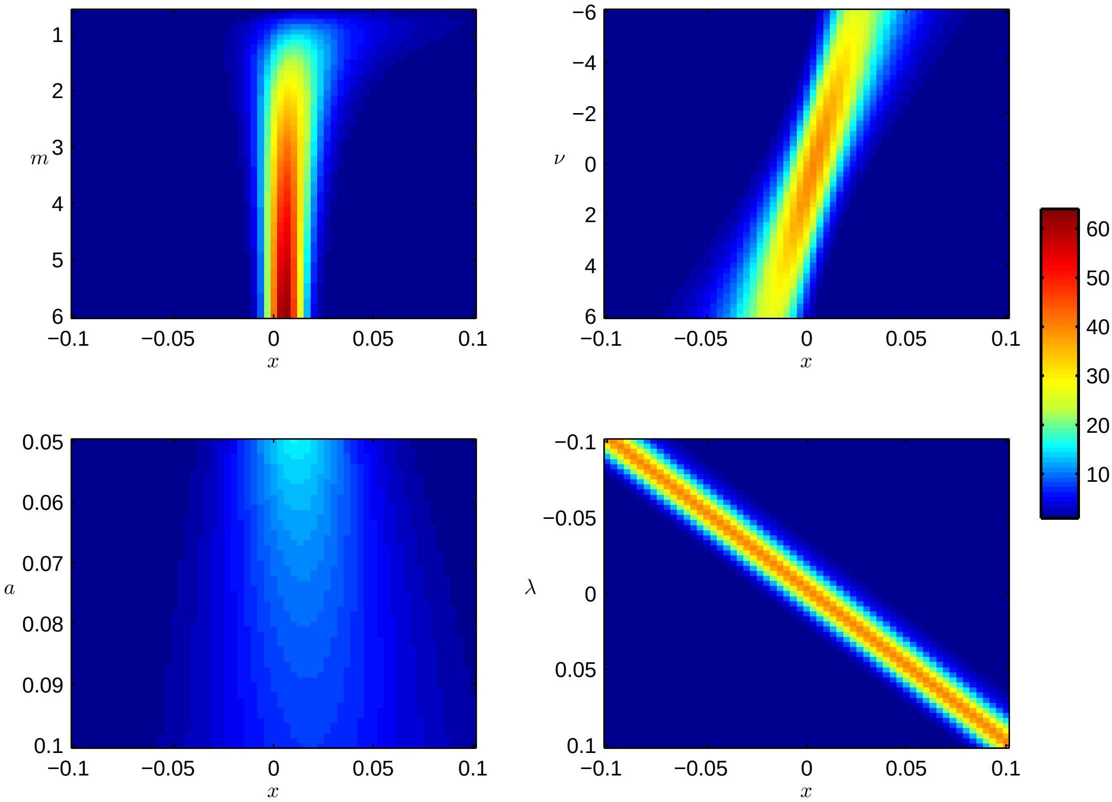

Figure 4.2: Image plots of the Pearson Type IV probability density function in which one parameter is varied and the remaining three are fixed. The colour bar indicates the value of the probability density function. When the parameters are fixed they are chosen as follows: $m = 2.8$, $\nu = -0.8$, $a = 0.02$ and $\lambda = 0.004$.

we have that

$$\begin{aligned} 1 - e^{2i\phi} &= 1 - \cos 2\phi - i \sin 2\phi \\ &= 1 - \cos 2\phi - 2i \sin \phi \cos \phi \\ &= 2 \sin^2 \phi - 2i \sin \phi \cos \phi \\ &= -2i(\cos \phi + i \sin \phi) \sin \phi \\ &= -2ie^{i\phi} \sin \phi. \end{aligned}$$

$\square$

Our next result uses the hypergeometric function.

Definition 1. The complex hypergeometric function is defined here as the analytic continuation of the hypergeometric series

$$\begin{aligned}&{}_2F_1(a,b;c;z)\\ =&1+\frac{ab}{c}z+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\ldots+\frac{a(a+1)\ldots(a+j-1)b(b+1)\ldots(b+j-1)}{c(c+1)\ldots(c+j-1)}\frac{z^j}{j!}+\ldots\end{aligned}$$

(Press et al., 1992, §5.14). The series is absolutely convergent for $|z|<1$, and has a singularity at $z=1$.

Proposition 2. For $0<y<\pi$ and $r>0$

$$\int_0^y e^{-\nu\phi}\sin^r\phi d\phi=\frac{e^{-\nu y}\sin^{r+1}y}{r+1}e^{iy}\,{}_2F_1\left(1,\frac{r+2+i\nu}{2};r+2;-2ie^{iy}\sin y\right). \tag{4.2}$$

Proof. First put $w=1-e^{2i\phi}=-2ie^{i\phi}\sin\phi$ from Lemma 2, then

$$e^{-\nu\phi}=(1-w)^{\frac{i\nu}{2}} \tag{4.3}$$

and

$$\sin^r\phi=\left(\tfrac{i}{2}\right)^r w^r e^{-ri\phi}=(-2i)^{-r}w^r(1-w)^{-\frac{r}{2}}. \tag{4.4}$$

Also,

$$dw=-2ie^{2i\phi}d\phi=-2i(1-w)d\phi \tag{4.5}$$

which implies that

$$d\phi=(-2i)^{-1}(1-w)^{-1}dw \tag{4.6}$$

Applying Lemma 2 we see that as ϕ ranges from zero to y, w ranges from zero to $z:=1-e^{2iy}$ and so

$$\int_0^y e^{-\nu\phi}\sin^r\phi d\phi=(-2i)^{-r-1}\int_0^z w^r(1-w)^{(i\nu-r-2)/2}dw. \tag{4.7}$$

Now (4.7) is an incomplete Beta function

$$\begin{aligned}\int_0^z w^{\alpha-1}(1-w)^{\beta-1}dw &= \frac{z^\alpha}{\alpha}\,{}_2F_1(\alpha,1-\beta;\alpha+1;z)\\ &= \frac{z^\alpha}{\alpha}\,{}_2F_1(1-\beta,\alpha;\alpha+1;z)\\ &= \frac{z^\alpha(1-z)^\beta}{\alpha}\,{}_2F_1(1,\alpha+\beta;\alpha+1;z).\end{aligned}$$

The first equality above is (Abramowitz and Stegun, 1974, 6.6.8) and the last is (Abramowitz and Stegun, 1974, 15.3.3). Now put $\alpha = r+1$ and $\beta = \frac{i\nu - r}{2}$. Since

$$z^\alpha = \left(-2ie^{iy}\sin y\right)^{r+1} = (-2i)^{r+1}e^{iy}e^{riy}\sin^{r+1} y$$

and

$$(1-z)^\beta = \left(1-\left(1-e^{2iy}\right)\right)^{\frac{i\nu-r}{2}} = e^{-\nu y}e^{-riy}$$

we have

$$\int_0^y e^{-\nu\phi}\sin^r\phi d\phi = \frac{e^{-\nu y}\sin^{r+1} y}{r+1}e^{iy}\ {}_2F_1\left(1, \frac{i\nu+r+2}{2}; r+2; -2ie^{iy}\sin y\right).$$

□

Theorem 2. The cumulative distribution function of the Pearson type IV distribution is given by

$$F(m,\nu,a,\lambda;x) = f(m,\nu,a,\lambda;x)\frac{a}{2m-1}\left(i - \frac{x-\lambda}{a}\right)\ {}_2F_1\left(1, m+i\frac{\nu}{2}; 2m; \frac{2}{1-i\frac{x-\lambda}{a}}\right). \quad (4.8)$$

Proof. Firstly, set $u = \frac{t-\lambda}{a}$

$$F(m,\nu,a,\lambda;x) = k(m,\nu,a)a\int_{-\infty}^{\frac{x-\lambda}{a}}(1+u^2)^{-m}e^{-\nu\arctan u}du.$$

Substituting u with $\tan\theta$ yields

$$F(m,\nu,a,\lambda;x) = k(m,\nu,a)a\int_{-\frac{\pi}{2}}^{\arctan\frac{x-\lambda}{a}}\cos^{2m-2}\theta e^{-\nu\theta}d\theta. \quad (4.9)$$

We continue by setting $\phi = \theta + \frac{\pi}{2}$, then

$$\begin{aligned} & F(m,\nu,a,\lambda;x) \\ = \ & k(m,\nu,a)ae^{\nu\frac{\pi}{2}}\int_0^{\arctan\frac{x-\lambda}{a}+\frac{\pi}{2}}\sin^{2m-2}\phi e^{-\nu\phi}d\phi \quad (4.10) \\ = \ & k(m,\nu,a)a\frac{e^{\nu\frac{\pi}{2}-\nu y+iy}\sin^{2m-1} y}{2m-1}\ {}_2F_1\left(1, m+i\frac{\nu}{2}; 2m; -2ie^{iy}\sin y\right) \quad (4.11) \end{aligned}$$

where the second equality is given by Proposition 2 and $y = \arctan\frac{x-\lambda}{a} + \frac{\pi}{2}$.
Now

$$e^{-\nu y} = e^{-\nu\left(\arctan\frac{x-\lambda}{a}+\frac{\pi}{2}\right)},$$

$$
\begin{aligned}
e^{iy} &= e^{i\left(\arctan\frac{x-\lambda}{a}+\frac{\pi}{2}\right)} \\
&= \cos\left(\arctan\frac{x-\lambda}{a}+\frac{\pi}{2}\right) + i\sin\left(\arctan\frac{x-\lambda}{a}+\frac{\pi}{2}\right) \\
&= -\sin\left(\arctan\frac{x-\lambda}{a}\right) + i\cos\left(\arctan\frac{x-\lambda}{a}\right) \\
&= -\frac{x-\lambda}{\sqrt{a^2+(x-\lambda)^2}} + i\frac{a}{\sqrt{a^2+(x-\lambda)^2}} \\
&= \frac{-(x-\lambda)+ia}{\sqrt{a^2+(x-\lambda)^2}}.
\end{aligned}
$$

In order to see how the second last equality was found, see Figure 4.3. Also,

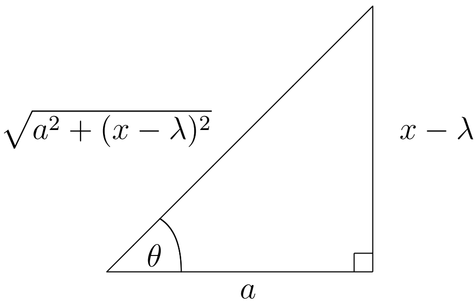

Figure 4.3: Triangle to show how to calculate trigonometric functions of $\arctan\left(\frac{x-\lambda}{a}\right)$.

$$
\begin{aligned}
\sin y &= \sin\left(\arctan\frac{x-\lambda}{a}+\frac{\pi}{2}\right) \\
&= \cos\left(\arctan\frac{x-\lambda}{a}\right) \\
&= \frac{a}{\sqrt{a^2+(x-\lambda)^2}}.
\end{aligned}
$$

Thus it follows that

$$
\begin{aligned}
-2ie^{iy}\sin y &= -2i\frac{-(x-\lambda)+ia}{\sqrt{a^2+(x-\lambda)^2}}\cdot\frac{a}{\sqrt{a^2+(x-\lambda)^2}} \\
&= \frac{2a^2+2a(x-\lambda)i}{a^2+(x-\lambda)^2} \\
&= \frac{2+2i\frac{x-\lambda}{a}}{1+\left(\frac{x-\lambda}{a}\right)^2} \\
&= 2\frac{1+i\frac{x-\lambda}{a}}{\left(1+i\frac{x-\lambda}{a}\right)\left(1-i\frac{x-\lambda}{a}\right)} \\
&= \frac{2}{1-i\frac{x-\lambda}{a}}.
\end{aligned}
$$

and the required result follows. □

4.2.1 Computational Issues

The cumulative distribution function given in (4.8) appears to be a complex function, but actually has imaginary part equal to zero. Matlab calculates ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ using a series expansion, the result is a complex number, so the value of $F(m,\nu,a,\lambda;x)$ returned will be in complex form; having used a series approximation it will include a very small imaginary part which is of the order 10^{-18}. In the code we simply set this imaginary part equal to zero.

As mentioned before, the hypergeometric series for ${}_2F_1(a,b;c;z)$ is only absolutely convergent for $|z|<1$, and has a singularity at $z=1$. When $z=\frac{2}{1-i\frac{x-\lambda}{a}}$ and $|z|<1$, we have

$$
\begin{aligned}
\left|\frac{2}{1-i\frac{x-\lambda}{a}}\right| < 1 \quad &\Rightarrow \quad \left|1-i\frac{x-\lambda}{a}\right| > 2 \\
&\Rightarrow \quad 1+\left(\frac{x-\lambda}{a}\right)^2 > 4 \\
&\Rightarrow \quad \left(\frac{x-\lambda}{a}\right)^2 > 3
\end{aligned}
$$

and so

$$
\begin{aligned}
\frac{x-\lambda}{a} < -\sqrt{3} \quad &\text{or} \quad \frac{x-\lambda}{a} > \sqrt{3} \\
\Rightarrow x < \lambda - a\sqrt{3} \quad &\text{or} \quad x > \lambda + a\sqrt{3}.
\end{aligned}
$$

Associated with the singularity at $z=1$ there is a branch cut, which could be taken to be any curve radiating out from $z=1$, and is only by convention chosen to lie along the real axis with

$\Re(z) \geq 1$. Thus, we are safe if $x < \lambda - a\sqrt{3}$. There is an interference with the branch cut if $x > \lambda + a\sqrt{3}$. Here it is simplest to apply the transformation

$$F(m, \nu, a, \lambda; x) = 1 - F(m, -\nu, a, -\lambda; -x). \tag{4.12}$$

When $|x - \lambda| < a\sqrt{3}$, one can transform $z \to \frac{1}{z}$ as in (Abramowitz and Stegun, 1974, 15.3.7). In this case it can be shown (see Heinrich (2004)) that

$$\begin{aligned} F(m, \nu, a, \lambda; x) &= f(m, \nu, a, \lambda; x) \frac{ia}{2m - i\nu - 2} \left[1 + \left(\frac{x - \lambda}{a} \right)^2 \right] \\ &\cdot {}_2F_1 \left(1, 2 - 2m; 2 - m + \frac{i\nu}{2}; \frac{1 + i\frac{x-\lambda}{a}}{2} \right) + \frac{1}{1 - e^{-\pi(\nu + 2im)}}. \end{aligned} \tag{4.13}$$

When programming in Matlab, the last transformation, (4.13) can be avoided entirely. This is done by considering only the two cases when $x < \lambda$ and $x > \lambda$. The amount of time taken in computing the cumulative distribution function when the last transformation is included, is only slightly faster than when we consider only the two cases and so it is not necessary to include the last transformation in the code. However, when programming in C++ this transformation must be included (see (Press et al., 1992, §6.12)). When we calculate the weights for the fund of funds through time, implementing the hypergeometric function in Matlab causes a problem with memory. This is because of a memory leak resulting after many calls to the Matlab function `maple()` in the Symbolic Math Toolbox[2]. Thus we have included a C++ dll for the cumulative normal distribution function of the Pearson Type IV distribution. Furthermore, this increases the speed of calculating the weights considerably.

4.2.2 Normalisation Constant $k(\cdot, \cdot, \cdot)$

Here we review the derivation of the normalisation constant by Nagahara (1999) and Heinrich (2004). We now derive the normalisation constant $k(\cdot, \cdot, \cdot)$ in (4.1)

$$k(m, \nu, a) = \frac{\Gamma(m)}{\sqrt{\pi} a \Gamma\left(m - \frac{1}{2}\right)} \left| \frac{\Gamma\left(m + i\frac{\nu}{2}\right)}{\Gamma(m)} \right|^2. \tag{4.14}$$

In (4.14), $\Gamma(\cdot)$ is the Gamma function. The Gamma function defined in terms of *Euler's integral* is

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \tag{4.15}$$

[2]One can find this discussed in several internet blogs by simply typing the words 'memory leak maple matlab' into a search engine.

for complex number z with $\Re(z) > 0$ as found in (Abramowitz and Stegun, 1974, 6.1.1). Using integration by parts (4.15) becomes

$$\begin{aligned} \Gamma(z) &= \left[-t^{z-1}e^{-t}\right]_0^\infty + (z-1)\int_0^\infty e^{-t}t^{z-2}dt \\ &= (z-1)\Gamma(z-1). \end{aligned} \tag{4.16}$$

From (4.10) we have that

$$1 = \int_{-\infty}^{\infty} f\left(m, \nu, a, \lambda; t\right) dt = k\left(m, \nu, a\right) a e^{\nu \frac{\pi}{2}} \int_0^\pi \sin^{2m-2}\phi e^{-\nu\phi} d\phi. \tag{4.17}$$

The following result, which was noted by Nagahara (1999), was proved by Nielsen (1906).

$$\int_0^\pi e^{-\nu\phi}\sin^r \phi d\phi = \frac{\pi\Gamma(x)^{-\frac{1}{2}\nu}}{2^r\Gamma\left(\frac{r+\nu i+2}{2}\right)\Gamma\left(\frac{r-\nu i+2}{2}\right)}.$$

Applying this result to (4.17) yields

$$\begin{aligned} 1 &= k\left(m,\nu,a\right) a e^{\nu\frac{\pi}{2}} \frac{\pi\Gamma\left(2m-1\right)e^{-\nu\frac{\pi}{2}}}{2^{2m-2}\Gamma\left(m+i\frac{\nu}{2}\right)\Gamma\left(m-i\frac{\nu}{2}\right)} \\ &= \frac{k\left(m,\nu,a\right) a\pi\Gamma\left(2m-1\right)}{2^{2m-2}\Gamma\left(m+i\frac{\nu}{2}\right)\Gamma\left(m-i\frac{\nu}{2}\right)}. \end{aligned}$$

And so

$$k(m,\nu,a) = \frac{2^{2m-2}\Gamma\left(m+i\frac{\nu}{2}\right)\Gamma\left(m-i\frac{\nu}{2}\right)}{a\pi\Gamma\left(2m-1\right)}. \tag{4.18}$$

Now, from (4.16) it follows that

$$\frac{\Gamma(2m)}{\Gamma(2m-1)} = 2m-1$$

and

$$\frac{\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m-\frac{1}{2}\right)} = m - \frac{1}{2}.$$

Also from (Abramowitz and Stegun, 1974, 6.1.18) we have the *duplication formula*

$$\Gamma(2m) = \frac{2^{2m-\frac{1}{2}}}{\sqrt{2\pi}}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right).$$

Therefore

$$\begin{aligned}\Gamma(2m-1) &= \frac{\Gamma(2m)}{2m-1}\\ &= \frac{2^{2m-\frac{1}{2}}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)}{\sqrt{2\pi}(2m-1)}\\ &= \frac{2^{2m-1}\Gamma(m)\frac{1}{2}(2m-1)\Gamma\left(m-\frac{1}{2}\right)}{\sqrt{\pi}(2m-1)}\\ &= \frac{2^{2m-2}\Gamma(m)\Gamma\left(m-\frac{1}{2}\right)}{\sqrt{\pi}}.\end{aligned}$$

Thus substituting $\Gamma(2m-1)$ into (4.18) yields

$$\begin{aligned}k(m,\nu,a) &= \frac{2^{2m-2}\Gamma\left(m+i\frac{\nu}{2}\right)\Gamma\left(m-i\frac{\nu}{2}\right)\sqrt{\pi}}{a\pi 2^{2m-2}\Gamma(m)\Gamma\left(m-\frac{1}{2}\right)}\\ &= \frac{\Gamma\left(m+i\frac{\nu}{2}\right)\Gamma\left(m-i\frac{\nu}{2}\right)}{a\sqrt{\pi}\Gamma(m)\Gamma\left(m-\frac{1}{2}\right)}\\ &= \frac{\Gamma\left(m+i\frac{\nu}{2}\right)\Gamma\left(m-i\frac{\nu}{2}\right)}{a\sqrt{\pi}\Gamma(m)\Gamma\left(m-\frac{1}{2}\right)}\cdot\frac{\Gamma(m)}{\Gamma(m)}\\ &= \frac{\Gamma(m)\Gamma\left(m+i\frac{\nu}{2}\right)\Gamma\left(m-i\frac{\nu}{2}\right)}{a\sqrt{\pi}\Gamma\left(m-\frac{1}{2}\right)\Gamma^2(m)}.\end{aligned}$$

So far we have been following Nagahara (1999). However, as noted by Heinrich (2004) this is not computationally straightforward. Heinrich (2004) performs some further manipulations which we give now. From (Abramowitz and Stegun, 1974, 6.1.23) we have that

$$\Gamma\left(\overline{z}\right) = \overline{\Gamma(z)}$$

where z is a complex number. It follows that

$$\Gamma\left(m+i\frac{\nu}{2}\right)\Gamma\left(m-i\frac{\nu}{2}\right) = \left|\Gamma\left(m+i\frac{\nu}{2}\right)\right|^2$$

and $k(m,\nu,a)$ is equal to

$$\frac{\Gamma(m)}{\sqrt{\pi}a\Gamma\left(m-\frac{1}{2}\right)}\left|\frac{\Gamma\left(m+i\frac{\nu}{2}\right)}{\Gamma(m)}\right|^2. \tag{4.19}$$

The Gamma function in Matlab does not take complex numbers as input. However, an algorithm exists to compute $\left|\frac{\Gamma\left(x+i\frac{y}{2}\right)}{\Gamma(x)}\right|^2$ as a function of the real inputs x, y - (Heinrich, 2004, §5.1) provides the C++ code for this. The Matlab code transcribed from the C++ code is given below

```
function gammar2Value = gammar2(x,y)
  ysquared = y^2;
  df = 0;
  r = 1;
  s = 1;
  p = 1;
  localx = x;
  epsilon = 10^(-16);

  if ysquared > 5
    xmin = 2*ysquared;
  else
    xmin = 10;
  end

  while xmin > localx
    t = y / localx;
    localx = localx + 1;
    r = r*(1 + t^2);
  end

  while p > s*epsilon
    p = p*(ysquared + df^2);
    p = p/(localx*(df + 1));
    localx = localx + 1;
    df = df + 1;
    s = s + p;
  end

  gammar2Value = 1/(r*s);

end
```

4.2.3 Cumulative Inverse Function

In Matlab, we calculate the cumulative inverse of the Pearson Type IV distribution using the function `fzero()` which finds the value of x_α for which the Pearson Type IV cumulative distribution function is equal to α. The syntax is given by

```
fzero(@(x) PearsonIVCDF(m,nu,a,lambda,x) - alpha,0)
```

Here `fzero()` finds the value of x for which the Pearson Type IV cumulative distribution function minus α is equal to 0. `fzero()` uses Brent's algorithm.

4.3 $\int_{-\infty}^{x} tf(m,\nu,a,\lambda;t)dt$ and $\int_{-\infty}^{x} t^2 f(m,\nu,a,\lambda;t)dt$

We find $\int_{-\infty}^{x} tf(m,\nu,a,\lambda;t)dt$ and $\int_{-\infty}^{x} t^2 f(m,\nu,a,\lambda;t)dt$ of a Pearson Type IV distributed variable. We will thus be able to find the mean and variance.

Theorem 3.

$$\int_{-\infty}^{x} tf(m,\nu,a,\lambda;t)dt = \left(\lambda - \frac{a\nu}{2m-2}\right) F(m,\nu,a,\lambda;x) - \frac{k(m,\nu,a)a^2 e^{-\nu \arctan \frac{x-\lambda}{a}}}{(2m-2)\left[1+\left(\frac{x-\lambda}{a}\right)^2\right]^{m-1}}. \quad (4.20)$$

Proof. First consider

$$\int_{-\infty}^{x} tf(m,\nu,a,\lambda;t)dt = k(m,\nu,a)\int_{-\infty}^{x} t\left[1+\left(\frac{t-\lambda}{a}\right)^2\right]^{-m} e^{-\nu \arctan \frac{t-\lambda}{a}}dt$$

As before, let $\theta = \arctan \frac{t-\lambda}{a}$ and so

$$\begin{aligned}
\int_{-\infty}^{x} tf(m,\nu,a,\lambda;t)dt &= k(m,\nu,a)\int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} (a\tan\theta+\lambda)\left(1+\tan^2\theta\right)^{-m} e^{-\nu\theta} a\sec^2\theta d\theta \\
&= k(m,\nu,a)\int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} a\,(a\tan\theta+\lambda)\sec^{-2m+2}\theta e^{-\nu\theta}d\theta \\
&= k(m,\nu,a)\int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} a\,(a\tan\theta+\lambda)\cos^{2m-2}\theta e^{-\nu\theta}d\theta \\
&= k(m,\nu,a)a^2\int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} \sin\theta\cos^{2m-3}\theta e^{-\nu\theta}d\theta \\
&\quad +\lambda k(m,\nu,a)a\int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} \cos^{2m-2}\theta e^{-\nu\theta}d\theta.
\end{aligned}$$

From (4.9) we have that

$$\lambda k(m,\nu,a)a\int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} \cos^{2m-2}\theta e^{-\nu\theta}d\theta = \lambda F(m,\nu,a,\lambda;x)$$

and

$$
\begin{aligned}
& k(m,\nu,a)a^2 \int_{-\frac{\pi}{2}}^{\arctan\frac{x-\lambda}{a}} \sin\theta\cos^{2m-3}\theta e^{-\nu\theta}d\theta \\
= \;& k(m,\nu,a)a^2 \left(\left[-\frac{1}{2m-2}\cos^{2m-2}\theta e^{-\nu\theta} \right]_{-\frac{\pi}{2}}^{\arctan\frac{x-\lambda}{a}} - \frac{\nu}{2m-2}\int_{-\frac{\pi}{2}}^{\arctan\frac{x-\lambda}{a}} \cos^{2m-2} e^{-\nu\theta}d\theta \right) \\
= \;& k(m,\nu,a)a^2 \left[-\frac{1}{2m-2}e^{-\nu\arctan\frac{x-\lambda}{a}} \left(\frac{a^2}{(x-\lambda)^2+a^2} \right)^{m-1} \right] \\
& -\frac{a\nu}{2m-2}k(m,\nu,a)a\int_{-\frac{\pi}{2}}^{\arctan\frac{x-\lambda}{a}} \cos^{2m-2}\theta e^{-\nu\theta}d\theta \\
= \;& -\frac{k(m,\nu,a)a^2e^{-\nu\arctan\frac{x-\lambda}{a}}}{(2m-2)\left[1+\left(\frac{x-\lambda}{a}\right)^2\right]^{m-1}} - \frac{a\nu}{2m-2}F(m,\nu,a,\lambda;x).
\end{aligned}
$$

□

We thus recover the following known result.

Corollary 1. Let X be a random variable from the Pearson Type IV distribution. Then

$$\mathbb{E}[X] = \lambda - \frac{a\nu}{2m-2}, \; m > 1.$$

Proof. As $x \to \infty$, $F(m,\nu,a,\lambda;x) \to 1$ since $F(\cdot,\cdot,\cdot;\cdot)$ is a cumulative distribution function and $\frac{1}{1+\left(\frac{x-\lambda}{a}\right)^2} \to 0$. Therefore

$$\mathbb{E}[X] = \lim_{x\to\infty}\int_{-\infty}^{x} tf(m,\nu,a,\lambda;t)dt = \int_{-\infty}^{\infty} tf(m,\nu,a,\lambda;t)dt = \lambda - \frac{a\nu}{2m-2}.$$

□

Theorem 4. For $m > \frac{3}{2}$,

$$
\begin{aligned}
& \int_{-\infty}^{x} t^2 f(m,\nu,a,\lambda;t)dt \\
&= \left(\lambda^2 - a^2 - \frac{a\lambda\nu}{m-1}\right)F(m,\nu,a,\lambda;x) - \frac{\lambda k(m,\nu,a)a^2e^{-\nu\arctan\frac{x-\lambda}{a}}}{(m-1)\left[1+\left(\frac{x-\lambda}{a}\right)^2\right]^{m-1}} \\
&\quad + \left[\frac{2m-2}{2m-3} + \frac{\nu^2}{(2m-3)(2m-2)}\right]a^2F(m-1,\nu,a,\lambda;x). \qquad (4.21)
\end{aligned}
$$

Proof. Again, by substituting $\theta = \arctan \frac{t-\lambda}{a}$ we have

$$
\begin{aligned}
& \int_{-\infty}^{x} t^2 f(m,\nu,a,\lambda;t)dt \\
= \quad & k(m,\nu,a) \int_{-\infty}^{x} t^2 \left[1 + \left(\frac{t-\lambda}{a}\right)^2\right]^{-m} \exp\left[-\nu \arctan \frac{t-\lambda}{a}\right] dt \\
= \quad & k(m,\nu,a)a \int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} (a\tan\theta + \lambda)^2 \cos^{2m-2}\theta e^{-\nu\theta} d\theta \\
= \quad & k(m,\nu,a)a \int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} \left(a^2\tan^2\theta + 2a\lambda\tan\theta + \lambda^2\right) \cos^{2m-2}\theta e^{-\nu\theta} d\theta \\
= \quad & k(m,\nu,a)a \int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} \left(a^2\sin^2\theta\cos^{2m-4}\theta + 2a\lambda\sin\theta\cos^{2m-3}\theta + \lambda^2\cos^{2m-2}\theta\right) e^{-\nu\theta} d\theta \\
= \quad & k(m,\nu,a)a \int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} \left(a^2(1-\cos^2\theta)\cos^{2m-4}\theta + 2a\lambda\sin\theta\cos^{2m-3}\theta + \lambda^2\cos^{2m-2}\theta\right) e^{-\nu\theta} d\theta \\
= \quad & k(m,\nu,a)a \int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} \left(a^2\cos^{2m-4}\theta + 2a\lambda\sin\theta\cos^{2m-3}\theta + (\lambda^2 - a^2)\cos^{2m-2}\theta\right) e^{-\nu\theta} d\theta
\end{aligned}
$$

The above equation is split into three integrals which we consider separately.

- We begin with the last integral. Using (4.9) we find that

$$
(\lambda^2 - a^2)k(m,\nu,a)a \int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} \cos^{2m-2}\theta e^{-\nu\theta} d\theta \quad = \quad (\lambda^2 - a^2)F(m,\nu,a,\lambda;x).
$$

- The second integral is calculated using integration by parts

$$
\begin{aligned}
& 2\lambda k(m,\nu,a)a^2 \int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} \sin\theta\cos^{2m-3}\theta e^{-\nu\theta} d\theta \\
= \quad & 2\lambda k(m,\nu,a)a^2 \left(\left[-\frac{\cos^{2m-2}\theta e^{-\nu\theta}}{2m-2}\right]_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} - \frac{\nu}{2m-2} \int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} \cos^{2m-2}\theta e^{-\nu\theta} d\theta \right) \\
= \quad & -\frac{2\lambda k(m,\nu,a)a^2 e^{-\nu \arctan \frac{x-\lambda}{a}}}{(2m-2)\left[1 + \left(\frac{x-\lambda}{a}\right)^2\right]^{m-1}} - \frac{2\lambda\nu k(m,\nu,a)a^2}{2m-2} \int_{-\frac{\pi}{2}}^{\arctan \frac{x-\lambda}{a}} \cos^{2m-2}\theta e^{-\nu\theta} d\theta \\
= \quad & -\frac{\lambda k(m,\nu,a)a^2 e^{-\nu \arctan \frac{x-\lambda}{a}}}{(m-1)\left[1 + \left(\frac{x-\lambda}{a}\right)^2\right]^{m-1}} - \frac{a\lambda\nu}{m-1} F(m,\nu,a,\lambda;x)
\end{aligned}
$$

  where again we used (4.9).

- Finally we look at the first integral and as before we use (4.9)

$$k(m,\nu,a)a^3\int_{-\frac{\pi}{2}}^{\arctan\frac{x-\lambda}{a}}\cos^{2m-4}\theta e^{-\nu\theta}d\theta \;=\; \frac{k(m,\nu,a)}{k(m-1,\nu,a)}a^2F(m-1,\nu,a,\lambda;x).$$

From (4.19) it follows that

$$\begin{aligned}\frac{k(m,\nu,a)}{k(m-1,\nu,a)} &= \frac{\Gamma(m)}{\sqrt{\pi}a\Gamma\left(m-\frac{1}{2}\right)}\left|\frac{\Gamma\left(m+i\frac{\nu}{2}\right)}{\Gamma(m)}\right|^2\cdot\frac{\sqrt{\pi}a\Gamma(m-\frac{3}{2})}{\Gamma(m-1)}\left|\frac{\Gamma(m-1)}{\Gamma\left(m-1+i\frac{\nu}{2}\right)}\right|^2\\ &= \frac{m-1}{m-\frac{3}{2}}\left|\frac{m-1+i\frac{\nu}{2}}{m-1}\right|^2\\ &= \frac{m-1}{m-\frac{3}{2}}\cdot\frac{(m-1)^2+\frac{\nu^2}{4}}{(m-1)^2}\\ &= \frac{(m-1)^2+\frac{\nu^2}{4}}{(m-\frac{3}{2})(m-1)}\\ &= \frac{m-1}{m-\frac{3}{2}}+\frac{\nu^2}{4(m-\frac{3}{2})(m-1)}\\ &= \frac{2m-2}{2m-3}+\frac{\nu^2}{(2m-3)(2m-2)}.\end{aligned}$$

Here we've made use of (4.16). Therefore

$$k(m,\nu,a)a^3\int_{-\frac{\pi}{2}}^{\arctan\frac{x-\lambda}{a}}\cos^{2m-4}\theta e^{-\nu\theta}d\theta = \left[\frac{2m-2}{2m-3}+\frac{\nu^2}{(2m-3)(2m-2)}\right]a^2F(m-1,\nu,a,\lambda;x).$$

□

Again we recover the known:

Corollary 2. Let X be a random variable from the Pearson type IV distribution.

$$\mathbb{V}\text{ar}[X] \;=\; \frac{a^2}{2m-3}\left[1+\frac{\nu^2}{4(m-1)^2}\right],\; m>\frac{3}{2}.$$

Proof. We first find $\mathbb{E}\left[X^2\right]$ by considering (4.21). If we let $x\to\infty$, then $F(m,\nu,a,\lambda;x)$ and $F(m-1,\nu,a,\lambda;x)$ tends to 1. We then have that

$$\begin{aligned}\mathbb{E}[X^2] &= \lim_{x\to\infty}\int_{-\infty}^{x}t^2f(m,\nu,a,\lambda;t)dt\\ &= \lambda^2-a^2-\frac{a\lambda\nu}{m-1}+\left[\frac{2m-2}{2m-3}+\frac{\nu^2}{(2m-3)(2m-2)}\right]a^2.\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{V}\text{ar}\,[X] &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\
&= \lambda^2 - a^2 - \frac{a\lambda\nu}{m-1} + \left[\frac{2m-2}{2m-3} + \frac{\nu^2}{(2m-3)(2m-2)}\right] a^2 - \left(\lambda - \frac{a\nu}{2m-2}\right)^2 \\
&= \frac{a^2}{2m-3}\left[-2m+3 - \frac{\nu^2(2m-3)}{(2m-2)^2} + 2m-2 + \frac{\nu^2}{2m-2}\right] \\
&= \frac{a^2}{2m-3}\left[1 + \frac{-\nu^2(2m-3)+\nu^2(2m-2)}{(2m-2)^2}\right] \\
&= \frac{a^2}{2m-3}\left[1 + \frac{-\nu^2(2m-3)+\nu^2(2m-2)}{(2m-2)^2}\right] \\
&= \frac{a^2}{2m-3}\left[1 + \frac{\nu^2}{4(m-1)^2}\right].
\end{aligned}$$

□

4.4 Fitting the Parameters

4.4.1 Moment Matching

Historically, distribution fitting was done using the method of moments. The explicit formulae for m, ν, a and λ found using moment matching can be found in (Heinrich, 2004, §8). However those formulae do not necessarily make sense (the parameter values are not necessarily real) for all input data sets.

Nowadays, distribution fitting is done using maximum likelihood. We can still make use of the results produced by the method of moments. When choosing arbitrary initial parameters, the finding of the maximum likelihood to be described in §4.4.2 may require many iterations before converging to the final parameter set. The number of iterations may be reduced by choosing the initial parameters equal to those found by using moment matching.

Nevertheless it is not crucial to set the initial estimates in this way. Furthermore some pathological cases were found where the moment matching estimates are actually poor seeds for the maximum likelihood method; simple typical parameters will be better.

4.4.2 Maximum Likelihood Estimation

When applying the method of maximum likelihood we estimate the parameters m, ν, a and λ that maximise the likelihood function $L(m, \nu, a, \lambda)$. Assume that $x_1, \ldots, x_n$ are independent and identically distributed values from the Pearson Type IV distribution. Then the likelihood function is given by

$$\begin{aligned} L(m,\nu,a,\lambda) &= \prod_{i=1}^{n} f(m,\nu,a,\lambda;x_i) \\ &= \prod_{i=1}^{n} k(m,\nu,a,\lambda)\left[1+\left(\frac{x_i-\lambda}{a}\right)^2\right]^{-m} e^{-\nu\arctan\frac{x_i-\lambda}{a}} \end{aligned}$$

It is standard here to realise that since the log-function is monotone, estimating the parameters which maximise the likelihood function is equivalent to estimating the parameters that maximise the log of the likelihood function. The log-likelihood function is given by

$$\begin{aligned} \ln L(m,\nu,a,\lambda) &= \sum_{i=1}^{n} \ln f(m,\nu,a,\lambda;x_i) \\ &= n\ln k(m,\nu,a,\lambda) - m\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i-\lambda}{a}\right)^2\right] - \nu\sum_{i=1}^{n}\arctan\frac{x_i-\lambda}{a}. \end{aligned}$$

When programming in Matlab, instead of maximising the log-likelihood function, we will minimise the negative of the log-likelihood function by applying the function `fminsearch()`. This function's syntax is given below and an explanation of the arguments follows

```
fminsearch(@LogLikelihood,InitialParameters,Options)
```

- `@Loglikelihood` is the negated log-likelihood function which is to be minimised.
- `InitialParameters` contains the initial guesses for the parameters m, ν, a and λ.
- The last parameter, `Options`, contains the optimisation options specified.

`fminsearch()` does not make use of numerical or analytic gradients. Instead it uses the Nelder-Mead simplex search method which is a direct search method. The Nelder-Mead method performs unconstrained non-linear optimisation. But two of the parameters in the log-likelihood function are constrained - m must be greater than a half and a must be strictly positive. These rectangular constraints are incorporated by setting the log-likelihood function equal to a very large negative number if $a \leq 0$ or $m \leq \frac{1}{2}$.

It was found that at the extremes the log-likelihood function is quite flat as seen in some of the cases in Figure 4.4; thus if a maximum is not found for a 'reasonable' parameter set, the value of at least one of the parameters may diverge to $+\infty$ or $-\infty$. The algorithm only terminates when the tolerance criteria of the search are met. In fact, the code might overflow before this event occurs.

Extensive (and frustrating) numerical testing found that a reasonably satisfactory solution to this problem was to constrain m to be reasonably small. Thus we set $m \leq 50$ in the same way that we set $m \geq \frac{1}{2}$.

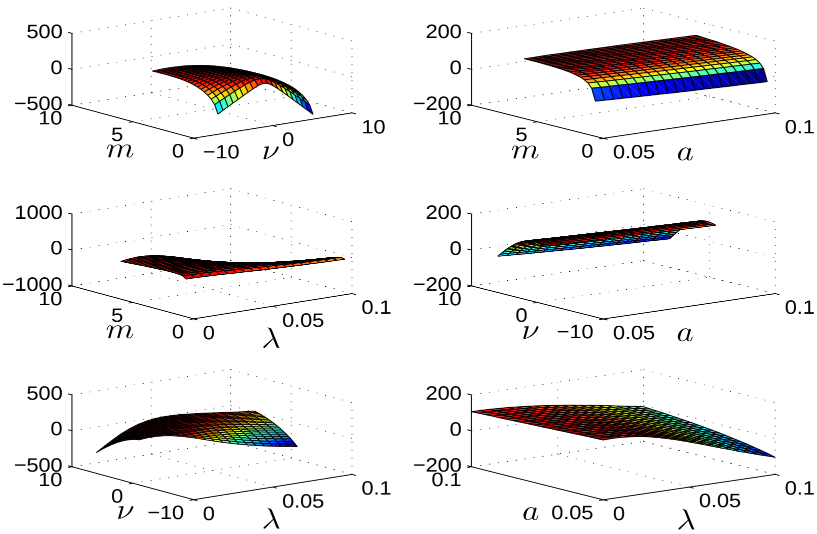

Figure 4.4: Surface plots of the Pearson Type IV log-likelihood function in which two parameters are varied and two are fixed. When the parameters are fixed they are chosen as follows: $m = 2.8$, $\nu = -0.8$, $a = 0.02$ and $\lambda = 0.004$.

In Figure 4.5 we have plotted histograms of six different indices of the EDHEC Alternative Indexes data along with their corresponding Pearson Type IV probability density functions with parameters m, ν, a and λ as determined by maximum likelihood estimation.

The diagram in Figure 4.6 illustrates how the optimal weights for time t_{i+1} are found using `fmincon()` when risk measures are calculated under the Pearson Type IV distribution. Here the optimal parameters are found using maximum likelihood estimation inside each iteration of `fmincon()`.

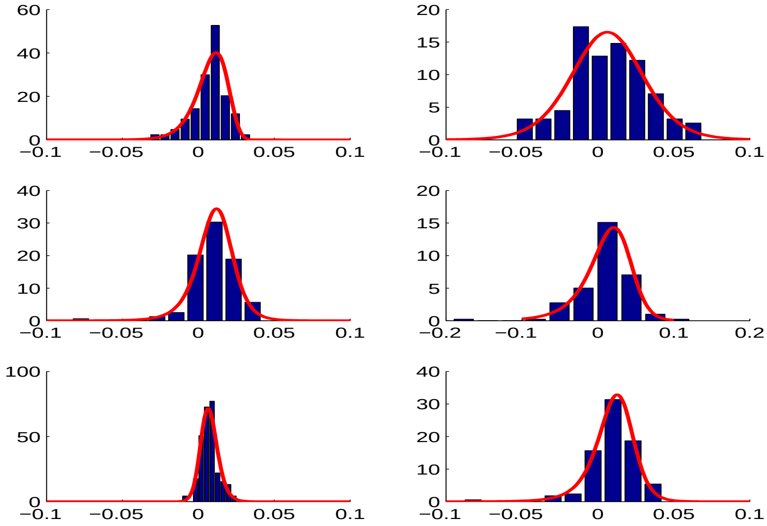

Figure 4.5: The Pearson Type IV probability density functions and their corresponding histograms for six different indices of the EDHEC Alternative Indexes data, EDHEC-Risk (2007).

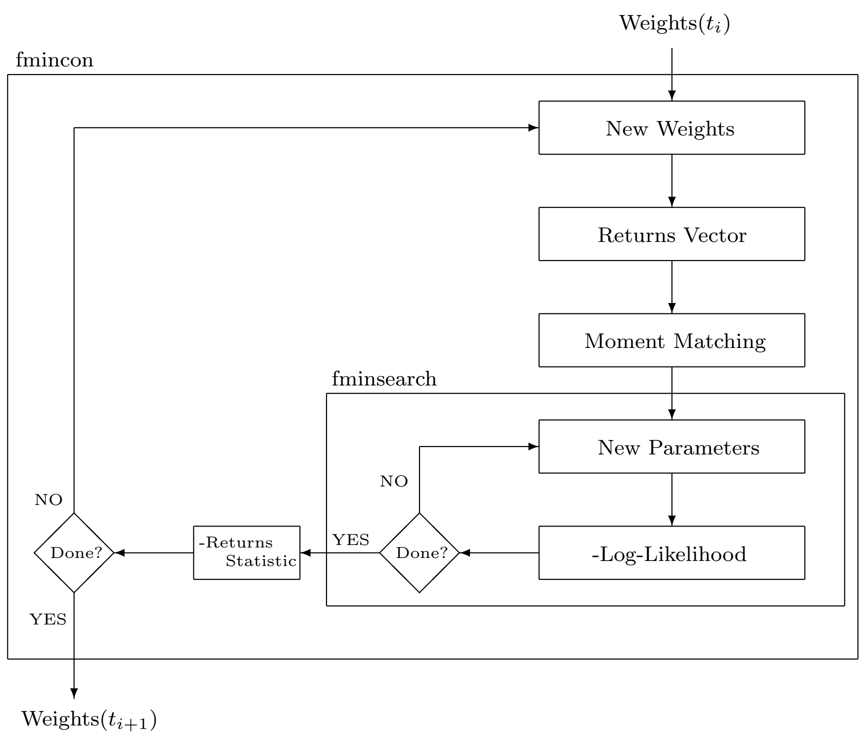

Figure 4.6: Diagram illustrating the method in which optimal weights are found for time t_{i+1}.

Chapter 5

Risk Measures using the Pearson Type IV Distribution

This chapter shows how to calculate the risk-adjusted measures Sharpe, Sortino and Omega as well as the risk measures Value-at-Risk and Expected Shortfall discussed in Chapter 2 using the Pearson Type IV distribution.

5.1 Sharpe

Recall the formula of the Sharpe risk-adjusted measure given by (2.1)

$$\text{Sharpe} = \frac{\mu - r}{\sigma}$$

where r is the risk-free return and μ and σ are the mean and standard deviation respectively of the Pearson Type IV distribution as given in Corollaries 1 and 2. Here, the parameter m must be greater than $\frac{3}{2}$.

5.2 Sortino

As given by (2.2) Sortino is calculated as

$$\text{Sortino } = \frac{\mu - r_{\text{MA}}}{\sigma_{\text{D}}}.$$

where r_{MA} indicates the minimum acceptable return, μ and σ_{D} are the mean and downside deviation respectively of the Pearson Type IV distribution. μ is given in Corollary 1. σ_{D} is determined by calculating downside deviation as follows

$$
\begin{aligned}
&\sqrt{\mathbb{V}\mathrm{ar}\left[X|X \le x\right]} \\
=&\sqrt{\mathbb{E}\left[X^2|X \le x\right] - \left(\mathbb{E}\left[X|X \le x\right]\right)^2} \\
=&\sqrt{\frac{\int_{-\infty}^{x} t^2 f(m,\nu,a,\lambda;t)dt}{\int_{-\infty}^{x} f(m,\nu,a,\lambda;t)dt} - \left(\frac{\int_{-\infty}^{x} tf(m,\nu,a,\lambda;t)dt}{\int_{-\infty}^{x} f(m,\nu,a,\lambda;t)dt}\right)^2} \\
=&\frac{\sqrt{\int_{-\infty}^{x} t^2 f(m,\nu,a,\lambda;t)dt \cdot \int_{-\infty}^{x} f(m,\nu,a,\lambda;t)dt - \left(\int_{-\infty}^{x} tf(m,\nu,a,\lambda;t)dt\right)^2}}{\int_{-\infty}^{x} f(m,\nu,a,\lambda;t)dt}. \qquad (5.1)
\end{aligned}
$$

Then using (5.1) we are able to calculate σ_{D} by substituting x with r_{MA}.
Thus it is possible to compute the Sortino ratio by using the formulae for $\int_{-\infty}^{r_{\mathrm{MA}}} tf(m,\nu,a,\lambda;t)dt$ given by Theorem 3 and $\int_{-\infty}^{r_{\mathrm{MA}}} t^2 f(m,\nu,a,\lambda;t)dt$ given by Theorem 4. Also $\int_{-\infty}^{r_{\mathrm{MA}}} f(m,\nu,a,\lambda;t)dt = F(m,\nu,a,\lambda;r_{\mathrm{MA}})$. Again, m must be greater than $\frac{3}{2}$.

5.3 Omega

The Omega risk-adjusted measure is given by (2.3) as

$$\Omega(L) = \frac{\int_{L}^{\infty} 1 - F(m,\nu,a,\lambda;t)dt}{\int_{-\infty}^{L} F(m,\nu,a,\lambda;t)dt}$$

where L is the chosen threshold level. In fact, it turns out that we can only define Omega for $m > 1$. This is not a problem because when fitting the data, the value of m is usually large. Using integration by parts

$$
\begin{aligned}
\int_{-\infty}^{L} F(m,\nu,a,\lambda;t)dt &= \left[tF(m,\nu,a,\lambda;t)\right]_{-\infty}^{L} - \int_{-\infty}^{L} tf(m,\nu,a,\lambda;t)dt \\
&= LF(m,\nu,a,\lambda;L) - \int_{-\infty}^{L} tf(m,\nu,a,\lambda;t)dt.
\end{aligned}
$$

This follows from the fact that

$$\begin{aligned}
\lim_{u\to-\infty} uF(m,\nu,a,\lambda;u) &= \lim_{u\to-\infty} \frac{F(m,\nu,a,\lambda;u)}{\frac{1}{u}} \\
&= \lim_{u\to-\infty} \frac{f(m,\nu,a,\lambda;u)}{-\frac{1}{u^2}} \text{ (by l'H\hat{o}pital)} \\
&= -\lim_{u\to-\infty} u^2 f(m,\nu,a,\lambda;u) \\
&= \lim_{u\to-\infty} u^2 k(m,\nu,a) \left[1+\left(\frac{u-\lambda}{a}\right)^2\right]^{-m} e^{-\nu \arctan\frac{u-\lambda}{a}} \\
&= 0
\end{aligned}$$

because $m > 1$. Also

$$\begin{aligned}
&\int_L^\infty 1 - F(m,\nu,a,\lambda;t)dt \\
&= [t\,(1-F(m,\nu,a,\lambda;t))]_L^\infty + \int_L^\infty tf(m,\nu,a,\lambda;t)dt \\
&= [tF(m,-\nu,a,-\lambda;-t)]_L^\infty + \int_L^\infty tf(m,\nu,a,\lambda;t)dt \\
&= -LF(m,-\nu,a,-\lambda;-L) + \int_{-\infty}^\infty tf(m,\nu,a,\lambda;t)dt - \int_{-\infty}^L tf(m,\nu,a,\lambda;t)dt \\
&= -LF(m,-\nu,a,-\lambda;-L) + \lambda - \frac{a\nu}{2m-2} - \int_{-\infty}^L tf(m,\nu,a,\lambda;t)dt
\end{aligned}$$

where the second equality follows from the transformation given by (4.12).
Figure 5.1 gives confirmation of the requirement that $m > 1$.

5.4 Value-at-Risk and Expected Shortfall

We find the x_α for which $F(m,\nu,a,\lambda;x_\alpha) = \alpha$. We have already discussed how this x_α is obtained using `fzero()` in §4.2.3. The Pearson Type IV Value-at-Risk is then $-V(t)x_\alpha$.
The Pearson Type IV Expected Shortfall is calculated as follows

$$\mathbb{E}[X|X \le x_\alpha] = \frac{\mathbb{E}[X\mathbb{1}_{\{X\le x_\alpha\}}]}{\mathbb{P}[X \le x_\alpha]} = \frac{1}{\alpha}\int_{-\infty}^{x_\alpha} tf(m,\nu,a,\lambda;t)dt \tag{5.2}$$

where the last equality can be calculated by substituting into (4.20). Just as in the case of Value-at-Risk, to find Expected Shortfall we now multiply this value by $-V(t)$.

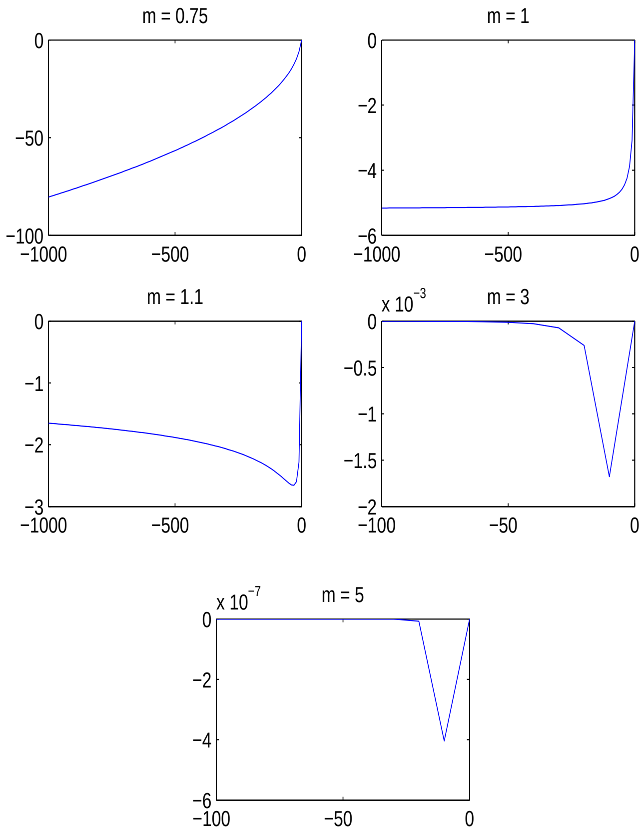

Figure 5.1: Pearson Type IV cumulative distribution function for different values of m.

5.4.1 Maximise Expected Returns to Expected Shortfall Constraint

Recall the general method for maximising a risk-adjusted measure in §3.1 using the `fmincon()` function in Matlab. Instead of maximising a risk-adjusted measure, we would like to maximise expected returns subject to the Expected Shortfall being less or equal to some risk limit. This constraint can be incorporated in the function `Constraint` which is the second last argument in the function `fmincon()`. The function `Constraint` will perform maximum likelihood estimation to calculate the parameters m, ν, a and λ. `Constraint` will then calculate $g(\mathbf{w})$ which is the Expected Shortfall at some point α minus the chosen risk limit. In this case there is no $h(\cdot)$ parameter, but it is compulsory to enter such a function into Matlab, so we set $h(\mathbf{w})$ to zero as a vacuous constraint.

Chapter 6

Results

In Figure 6.1 we graph the different portfolio weights for different threshold levels (Omega) and minimum acceptable returns (Sortino). This demonstrates that dramatically different portfolio weights are possible when varying the threshold level and the minimum acceptable return. It follows that reports that say the portfolio weights were chosen using the Omega or Sortino measures are almost meaningless, unless the threshold level or minimum acceptable return is specified, and there is some rational reason for that choice of threshold level or minimum acceptable return. Moreover, the threshold level or minimum acceptable return need to be set in advance of the analysis, not as a consequence of them.

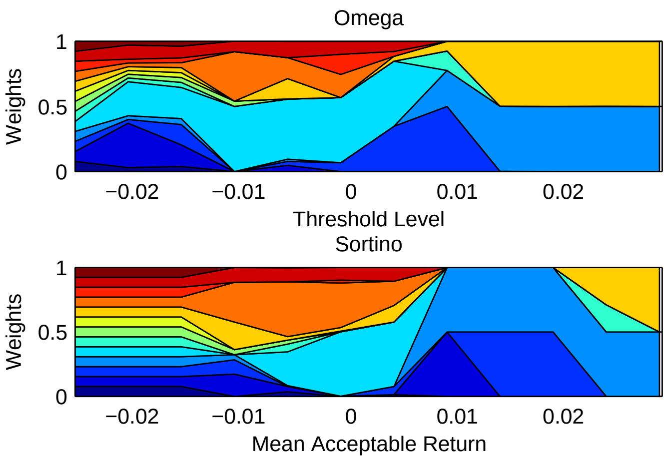

Figure 6.1: The portfolio selection weights determined using the empirical method and when varying the threshold level (Omega) and the mean acceptable return (Sortino).

Notwithstanding the concerns raised already with respect to Figure 6.1, in Figure 6.2 we set the

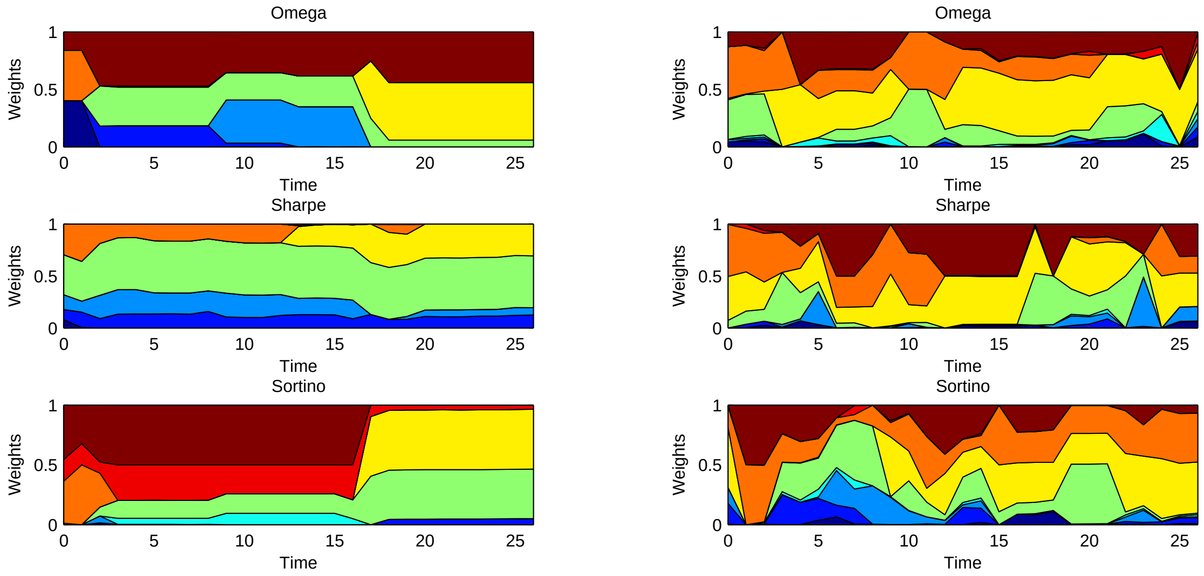

Figure 6.2: Comparing the portfolio weights of the empirical (left) and Pearson Type IV (right) risk-adjusted measures Omega, Sharpe and Sortino rebalanced monthly. The data used here is South African long-short hedge funds.

threshold level, the mean acceptable return and the risk-free return to 0.006 (all these levels are monthly). So these are the monthly rebalanced weights, starting at the halfway point in the data.

As discussed in §3.3, Figures 6.3 and 6.4 shows that rebalancing the portfolio weights through time only slightly improves the performance of the portfolio and in some cases (as seen in Figure 6.3 for the Sortino risk-adjusted measure) the portfolio performs worse. Figure 6.3 shows the results for empricially calculated measures and Figure 6.4 the results of the Pearson Type IV calculated measures.

In Figure 6.5 the negative Value-at-Risk and negative Expected Shortfall are plotted along with profit and losses realised for two funds for the time period where that risk measure applies. These measures are found using the Pearson Type IV distribution and the method in §2.4. These risk measures are price-measures, not return-measures. Note in the left figure even with a cluster of negative returns, Value-at-Risk or Expected Shortfall can be trending downwards. This is because the losses have caused the size of the portfolio to be decreased. Similarly, for the figure on the right, the Value-at-Risk and Expected Shortfall are increasing on average because of an average increase in the portfolio value.

In Figure 6.6, the results of the expected returns maximised subject to the Expected Shortfall

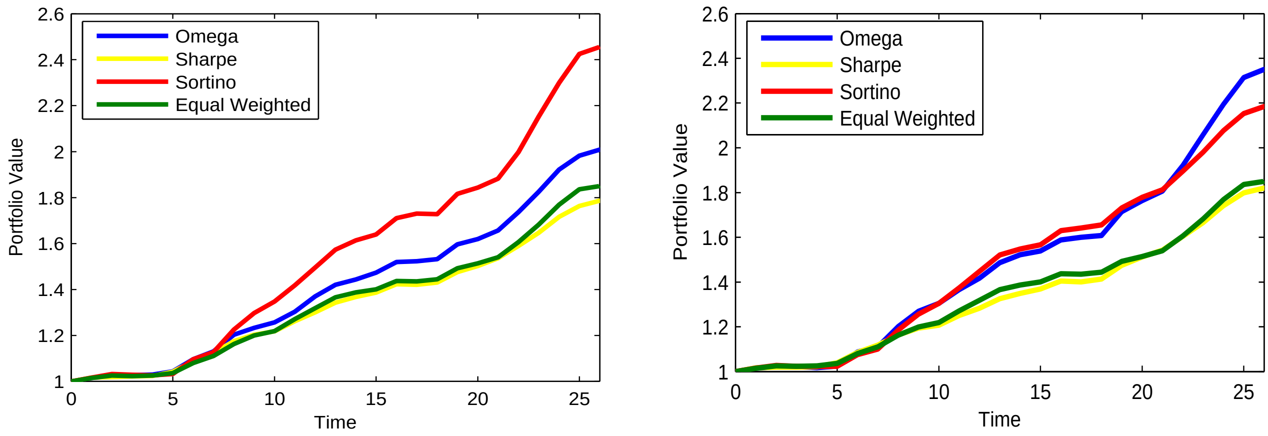

Figure 6.3: Comparing the portfolio values through time of the portfolio which has not been rebalanced (left) and the rebalanced portfolio (right). The data used here is South African long-short hedge funds. These portfolios were chosen by maximising the various empirical risk-adjusted measures.

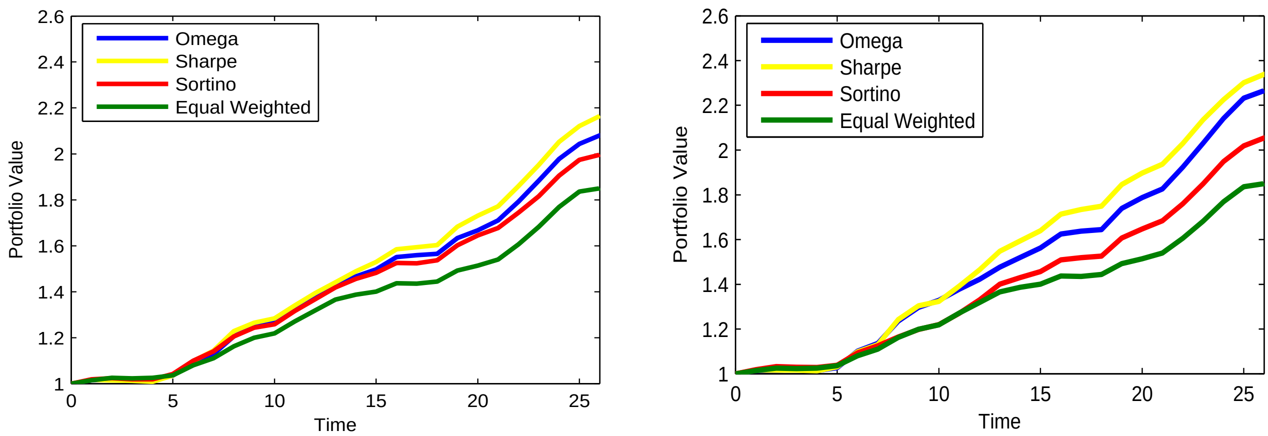

Figure 6.4: Comparing the portfolio values through time of the portfolio which has not been rebalanced (left) and the rebalanced portfolio (right). The data used here is South African long-short hedge funds. These portfolios were chosen by maximising the various Pearson Type IV risk-adjusted measures.

being less than some risk limit is shown. The method for this implementation was discussed in §5.4.1. Note the choice of weights are somewhat unstable.

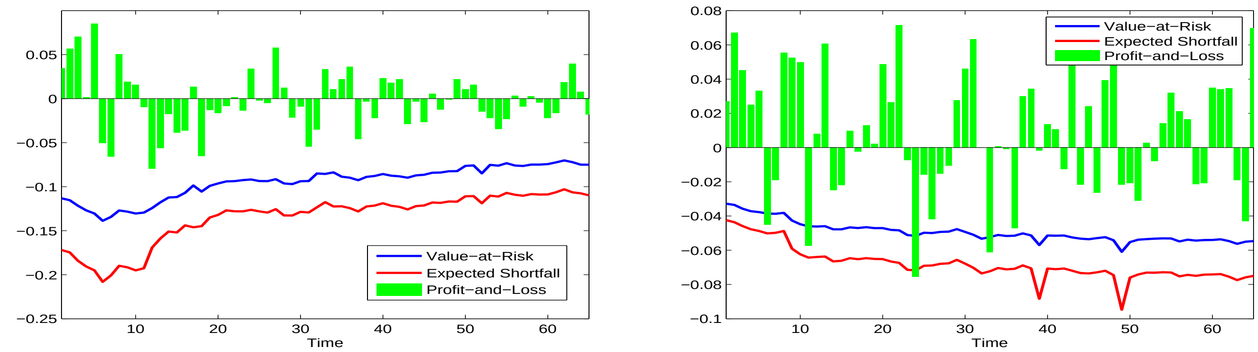

Figure 6.5: Value-at-risk and Expected Shortfall and the profit and losses of a portfolio under Pearson Type IV.

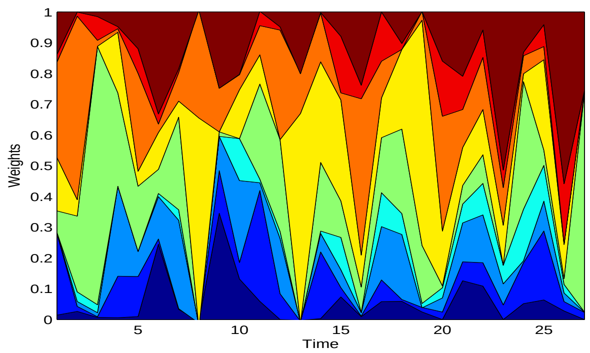

Figure 6.6: The portfolio weights through time of South African long-short hedge funds where the expected returns are maximised subject to the constraint that the Expected Shortfall is less than 0.01.

Chapter 7

Conclusion

In Chapter 6 it is shown that different parameters or assumptions may lead to very different results. Thus when interpreting hedge fund reports, one must be well aware of the assumptions made and parameters used. As Lhabitant (2004) states "Not surprisingly, for some years, unscrupulous product marketers have taken advantage of this difficulty. They simply considered hedge fund performance measurement as a game, following one guiding principle: 'Give me a fund and I will find the performance measure and the time period that makes it look attractive.' "

South African hedge fund data does not have a long history and even international data has few data points. This is because hedge fund data is reported monthly. Furthermore, to obtain large hedge fund data sets one has to pay prohibitive fees.

The different results obtained using the empirical method and the Pearson Type IV distribution to calculate the various risk measures can be attributed to this sparsity of data. It is more desirable to fit sparse data to a distribution than perform empirical calculations from it. Hedge funds are known for their extreme returns and thus the Pearson Type IV distribution is a good choice as distribution since it has the ability to fit data with excess kurtosis and skewness.

Another benefit of using a distributional assumption is that we are then able to compute Value-at-Risk and Expected Shortfall which is not possible to do otherwise.

Appendix A

Code

Matlab code to implement all the calculations in this project is provided, along with the data used for the analysis and the C++ for the Pearson Type IV distribution discussed in §4.2.1. In Table A.1 a list of the relevant m-files for the corresponding section are provided.

Section	Matlab Code
2.1	EmpiricalSharpe.m
2.2	EmpiricalSortino.m
2.3.1	EmpiricialOmega.m
2.3.2	EmpiricalOmega2FundsGraph.m (Figure 2.5)
4.1	PearsonIVPDF.m, PearsonIVPDFGraphs.m (Figure 4.1),
	PearsonIVPDFGraphs.m (Figure 4.2)
4.2	PearsonIVCDF.m
4.2.1	CPlusPlusPearsonIVCDF.m
4.2.2	PearsonIVNormalisationConstant.m, gammar2.m
4.2.3	PearsonIVCDFInverse.m
4.3	Integralxfdx.m, PearsonIVMean.m, Integralx2fdx.m, PearsonIVVariance.m
4.4.2	LogLikelihood.m, MaximumLikelihood.m
	PearsonIVLogLikelihoodGraphs.m (Figure 4.4)
	PearsonIVPDFandEmpiricalPMFGraphs.m (4.5)
5.1	PearsonIVSharpe.m
5.2	PearsonIVSortino.m
5.3	PearsonIVOmega.m, PearsonIVCDFmValuesGraphs.m (Figure 5.1)
5.4	PearsonIVVaR.m, PearsonIVES.m
6	EmpiricalOptimisationforDifferentLevels.m (Figure 6.1)
	EmpiricalOptimisation.m, PearsonIVOptimisation.m, MaximisePearsonIVSharpe.m,
	MaximisePearsonIVSortino.m, MaximisePearsonIVOmega.m (Figures 6.2,6.3,6.4)
	PearsonIVVARESGraph.m (Figure 6.5), PearsonIVEOptimisation.m,
	MaximisePearsonIVMean.m (Figure 6.6)

Table A.1: M-files associated with each section.

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