BOOTSTRAP ALGORITHM

GRAEME WEST, FINANCIAL MODELLING AGENCY

We present and clarify here the algorithm for the bootstrap procedure discussed in [Hagan and West, 2006, p.93] and [Hagan and West, 2008, §2]. Mechanisms for including OISs are in a separate document.

1. Algorithm for a swap curve bootstrap

The instruments available are LIBOR-type instruments, FRAs, and swaps. The assumption is that the LIBOR instruments and FRAs expire before the swaps.

(i) Arrange the instruments in order of expiry term in years. Let these terms be $\tau_1,\ \tau_2,\ \ldots,\ \tau_N$.

(ii) Find the NACC rates corresponding to the LIBOR instruments: suppose for the i^{th} instrument the rate quoted is L_i. Then

$$r_i = \frac{1}{\tau_i}\ln(1 + L_i\tau_i) \tag{1}$$

(iii) Now create a first estimate curve. For example, for each of the FRA and swap instruments let r_i be the market rate of the instrument.

(iv) (*) We now have a first estimate of our curve: terms $\tau_1,\ \tau_2,\ \ldots,\ \tau_M$ and rates $r_1,\ r_2,\ \ldots,\ r_M$. These values are passed to the interpolator algorithm.

(v) Update FRA estimates: when a FRA is dealt, for it to have zero value, we must have

$$C(t, t_1)(1 + \alpha f) = C(t, t_2) \tag{2}$$

Here t_1 is the settlement date of the FRA, t_2 the expiry date (thus, it is a $t_1 \times t_2$ FRA), f the FRA rate, and $\alpha = t_2 - t_1$ the period of the FRA, taking into account the relevant day count convention. We can rewrite (2) as

$$r(t_2) = \frac{1}{\tau_2}\left[C(t, t_1) + \ln(1 + \alpha f)\right] \tag{3}$$

and this gives us the required iterative formula for bootstrap: $C(t, t_1)$ on the right is found by reading off (interpolating!) off the estimated curve, the term that emerges on the left is noted for the next iteration.

(vi) When a swap is dealt, for it to have zero value, we have

$$R_n \sum_{i=1}^{n} \alpha_i Z(t, t_i) = 1 - Z(t, t_n)$$

and hence

$$Z(t, t_n) = \frac{1 - R_n \sum_{i=1}^{n-1} \alpha_i Z(t, t_i)}{1 + R_n \alpha_n} \tag{4}$$

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We can rewrite (4) as

$$r(t_n) = \frac{-1}{\tau_n} \ln \left[\frac{1 - R_n \sum_{j=1}^{n-1} \alpha_j Z(t, t_j)}{1 + R_n \alpha_n} \right] \tag{5}$$

and this gives us the required iterative formula for bootstrap: again all terms on the right are found by reading off (interpolating!) on the estimated curve, the term that emerges on the left is noted for the next iteration.

(vii) We do this for all instruments. We now return to (*) and iterate until convergence. Convergence to double precision is recommended; this will occur in about 10-20 iterations for favoured methods (monotone convex, raw).

2. Algorithm for a bond curve bootstrap

For the bootstrap of a South African bond curve, the instruments available are LIBOR instruments, FRAs, and bonds. Although this might appear irregular - mixing creditworthiness - it is standard practice. The same algorithm works elsewhere; one just deems the set of LIBOR and FRA instruments to be empty.

(i) The first steps (LIBOR, FRA) are as for the swap curve.

(ii) When it comes to the creation of the first provisional curve, we require an estimate for the NACC spot rate for the maturity of the bonds. Let r_i be the yield to maturity of the bond if the bond trades on yield (as in South Africa), or the NACC equivalent of that yield. If the yield to maturity is not available, any approximate value will do: for example, the last LIBOR or FRA rate as a flat estimate for all bonds might be good enough.

(iii) (*) We now have a estimate of our curve: terms $\tau_1, \tau_2, \ldots, \tau_M$ and rates $r_1, r_2, \ldots, r_M$. These values are passed to the interpolator algorithm.

(iv) We will consider two different ways of realising value for each bond:

$$[\mathbb{A}] Z(t, t_{\text{ssd}}) = \sum_{i=0}^{N} p_i Z(t, t_i) \tag{6}$$

where, for each bond,

$$\begin{aligned}
[\mathbb{A}] &= \text{rounded price of the bond} \\
\text{ssd} &= \text{the standard settlement period for bonds} \\
p_0 &= e\frac{c}{2} \\
p_1,\ p_2,\ \ldots,\ p_{N-1} &= \frac{c}{2} \\
p_N &= 1 + \frac{c}{2}{}^{[1]} \\
N &= \text{The } N \text{ in the bond pricing formula} \\
t_i &= \text{date of the coupon flow } i^{[2]}
\end{aligned}$$

Note that this is the critical equation of value: whether we realise the value in the bond market, or we strip the cash flows one at a time, we must (in principle) get the same value.

[1] In the case that $N = 0$ we have $p_N = 1 + e\frac{c}{2}$ and the equation that will follow solves immediately, without iteration.

[2] Note that this is found using the regular following date of the the designated coupon date.

The one equation has two unknowns. What we now do is impose a model - we make r_N the subject of the equation. Hence

$$r(t_N) = -\frac{1}{\tau_N}\left[\ln\left([\mathbb{A}]Z(t,t_{\text{ssd}}) - \sum_{i=0}^{N-1} p_i Z(t,t_i)\right) - \ln p_N\right] \tag{7}$$

Now, this equation is somewhat self-referential - it is an under-specified system - because the $r(t_i)$ are previously estimated by interpolation, which involves the given estimate of $r(t_N)$. This observation, however, sets up the iterative procedure, which may coincide with the simultaneous bootstrap method suggested in Smit and van Niekerk [1997].

We do this for all the instruments. We now return to (*) and iterate until convergence. Convergence to double precision is recommended; this will occur in about 10-20 iterations.

Bibliography

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Financial Modelling Agency, 19 First Ave East, Parktown North, 2193, South Africa.

E-mail address: `graeme@finmod.co.za`