A FINITE DIFFERENCE MODEL FOR VALUATION OF EMPLOYEE STOCK OPTIONS

GRAEME WEST, FINANCIAL MODELLING AGENCY

Abstract

Employee stock option grants are a common incentive for employees and are a key remuneration device. These options differ from ordinary options in that they cannot be traded nor hedged. Nevertheless, the work of Carpenter [1998] enables one to price these options within a Black-Scholes framework, with one additional parameter calibrated from historical data. We develop a model where the price of the grant obeys the Black-Scholes differential equation with two additional parameters: one which controls the rate at which employees forfeit unvested options, and another which controls the rate at which employees exercise vested options. We implement a finite difference scheme for computation of the option values derived from this model.

Keywords: Employee stock option, early exercise, finite difference scheme.

1. Introduction

Until the birth of option pricing algorithms in the 1970s the common wisdom concerning ESOs was that they were not an expense because there were no cash flow implications for the firm. The first valuation methodology to be introduced was that options should be expensed at their intrinsic value on grant date. Provided that these options were granted at the money, which was typical, there would be no profit and loss or bottom line effect.

In the 1990's it was recognised that options, even those struck at the money, had economic value, and thus needed to be expensed. However, very few companies chose to record any expenses for their ESOs, especially when there was no legal or regulatory requirement for such accounting treatment. In the USA for example, the Financial Accounting Standards Board only made footnote disclosure of such expenses mandatory, and because of this many companies did not record the impact of their ESO's, further than referring to it in a footnote to their financial statements.

After the dotcom crash, it was realised just how much of a burden ESOs were to the company. Many dotcom companies had relied very heavily on ESOs to incentivise employees, and these companies would have looked unprofitable a long time before they actually did, if those ESOs had been expensed, rather than just appear as footnotes to the accounts.

It is now almost universally accepted that ESOs are an expense to the company that need to be accounted for - see Bodie et al. [2003]. Audit requirements for ESOs fall under IFRS2 IASCF [2006] regulations. The company

Date: January 14, 2009.

is buying services from the employee; the remuneration is share-based. As such this remuneration needs to be expensed using a recognised option pricing formula.

It is important to note that the value of the asset that the individual employee stock option holder has is not equal to the liability of the company. This is because the employee has various constraints pertaining to his/her holding as opposed to an ordinary stock option held by an outsider. For instance, the employee is unable to sell the option on, and is unable to hedge it, as insiders are prevented from short selling the stock. As a result, an employee's valuation reflects risk aversion and other necessarily subjective preferences. In order to diversify wealth the option holder is willing to sacrifice value.

However, as enunciated for example in a discussion memo of the FASB dated December 15, 2003, the liability that the company needs to expense is equal to the amount they would be required to pay a hypothetical market participant to assume the employee stock option obligations.

For forfeitures, it is permitted to find the valuation of the option assuming no forfeitures, and then simply post-multiply by the expected proportion of grants that will survive to vesting date. This factor is estimated simply by considering a history of resignations from the company. This is the default approach specified in [IASCF, 2006, BC176], although it does allow the company to apply a more sophisticated approach. Of course this default assumption does not take into account any non-linearity in the hazard phenomenon - it is assuming that forfeiture is independent of the stock price process followed in the vesting period.

An enormous body of literature has built up on the valuation of these options, with many quite distinct approaches to valuation having been advocated. Our model is a natural extension of the seminal work of Carpenter [1998], which allows the Cox et al. [1979] framework to be used, with an additional 'abstract' factor which is an early termination (exercise or forfeiture) factor. This factor is calibrated to market data.

The approach we take generalises the work of Carpenter [1998] in two directions: firstly, use is made of a finite difference scheme rather than a binomial tree. This allows for the use of a term structure of interest rates, discrete dividend payments, and even in principle a local volatility surface. Secondly, we explicitly model the termination as due to early exercise (after vesting) or due to forfeiture (before vesting, and after vesting when out the money), and in principle these factors can separately be calibrated from the history of the scheme or from other like schemes. The model we develop allows for user input of those early exercise functions and forfeiture functions. Any such functions are permissible providing they are Markovian in nature, that is, they are determined by the state of the relevant variables, but not their history.

2. The structure of ESOs

Employee or executive stock options (ESOs for short) are call options granted by a company to an employee on the stock of the company. These options are part of the remuneration package of the employee. However, they differ from ordinary options in at least one crucial way: they cannot be transferred, and in the event of the employee leaving the company, they are forfeited.

They are a sweetener for the employee: they encourage him or her to remain an employee, and they encourage him or her to work towards an increase in the financial health of the company, which will translate into an increased share price, and eventual increased wealth of the employee.

Typically employee stock option grants are Bermudan options, where, after some vesting period, the employee may until some 'drop-dead' date exercise the options and take possession of the stock, or possibly receive cash. After vesting there may also be some closed periods, where exercise is not allowed. These will typically be in the period or periods before year-end or half-year results announcements. This may constitute as much as five months of the year, for example. Option holders will also be prevented from exercising if the company issues a cautionary notice to the market.

Typically, the vesting of the options proceeds in proportions. For example, an employee might have 40% of their options vest after 3 years, 30% after 4 years, and the last 30% after 5 years. To handle this one simply splits the analysis up into 3 separate grants, and aggregates at the end.

In reality, the employee may leave the employ of the firm at any time. In this case, any benefits of unvested options are voided.

Broadly, the rationale for vesting, and in particular proportional vesting, is in order to 'lock in' the employee in the service of the company for as long as possible. Hence employee stock options are the so-called 'golden handcuffs': they incentivise the employee to remain in the employ of the company, and as a group for the employees to work towards the best interests of the company (this assumes that the free-rider and other game-theoretic issues are not important).

3. Valuation

The point of view we take is that of the employer.

At first blush the option grants appear to be ordinary Bermudan options. However, additional factors are that the option may be forfeited (because of resignation of the employee) or exercised early, in such a way which is sub-optimal from the usual option pricing point of view.

The question of whether or not an employee will exercise according to usual option pricing criteria, or will exercise early, is very relevant. Within the risk-neutral pricing world the early exercise is sub-optimal behaviour, and is typically theoretically arbitragable. However, in this case, the employee is unable to hedge their position, and so risk-neutrality does not apply. Typically they will be risk averse, and so may exercise early. They may also exercise early in order to achieve additional liquidity in their personal wealth portfolio.

In Carpenter [1998] pricing options was performed using the usual risk-neutral valuation methodology, via a tree in the spirit of Cox et al. [1979], but with a single additional hazard factor which models early termination, which could be because

- the option has not vested, or has vested and is out the money, and the employee resigns (forfeiture);
- the option has vested and is in the money, and the employee exercises the option- notwithstanding the loss of time value (early exercise).

That is is possible to include this factor but still work in a risk-neutral world is argued in [Carpenter, 1998, §3.2].

The additional factor is an annualised rate of termination, due to either of the above reasons, and is denoted μ. The methodology for the valuation of these options will start by using a tree of share prices constructed in the standard way: constructing the binomial tree of stock prices using the classical method of Cox et al. [1979]. This methodology is compliant with the requirements outlined in IFRS2 for employee stock grant valuations. The tree is constructed from valuation date to the very last exercise date.

The model of Carpenter [1998] is to build the binomial tree of valuations, where at every branch we find the maximum of

- the value of exercising immediately (if legal, else this step is omitted);
- the value of holding to the next period, which is the discounted weighted average of the following four events:
  - the risk neutral value of the position if an up movement occurs, and termination does not occur, this occurring with probability $p(1-\mu dt)$, where p is the risk neutral probability of an up move;
  - the value of the position if an up movement occurs and termination then occurs, this occurring with probability $p\mu dt$;
  - the risk neutral value of the position if a down movement occurs, and termination does not occur, this occurring with probability $(1-p)(1-\mu dt)$;
  - the value of the position if a down movement occurs and termination then occurs, this occurring with probability $(1-p)\mu dt$.

A first generalisation of this model will be to have two additional factors rather than just one:

- a factor ν that models the per annum intensity of terminations to the grant caused by termination of employment (applies to grants before vesting, or after vesting but out of the money, or after vesting but during a closed period).
- another factor λ that models the per annum intensity of terminations to the grant caused by early exercise (applies to grants after vesting and in the money).

Both of these hazard rates can separately be estimated from historical data. The annual rate of resignations of employees in the categories of employee receiving grants is easily measured. Furthermore, the rate of grants being exercised after vesting can be measured and an annual hazard rate estimated statistically.

We will go one step further, as for example seen in the setting of the pricing of convertible bonds Ayache et al. [2002], Ayache et al. [2003] where the rate of default on convertible debt is functionally related to the stock price. Let the annualised rate of early exercise be $\lambda(S,t)$, that is, dependent on the stock price and on time. One wants a functional form with at least the following properties:

- $\lambda(S,t) = 0$ for $S < K$, or if it is a closed period.
- $\lambda(S,t) \uparrow$ as $S \uparrow,\ \sigma \downarrow,\ t \uparrow$.

According to [IASCF, 2006, B18] these and other factors such as peer scheme information and employee seniority are relevant. Klein and Maug [October 14, 2008] analyse the statistical significance of many parameters, many (but not all) of which we can incorporate in our hazard function.

Similarly let the annual forfeiture rate be $\nu(S,t)$. Of course where $\lambda(S,t) \neq 0$ we have $\nu(S,t) = 0$. Furthermore, if the option has vested and is in the money, but it is a closed period, $\nu(S,t)$ will be very small indeed, practically 0. The reasoning is simple: closed periods are typically not very long, and it seems unlikely that an employee would rather resign and forfeit their options than wait for a short period, exercise, and then resign.

For the moment we set the functional form for $\nu(S,t)$ to be 0 as above, and flat where it is not 0.

4. A finite difference implementation

4.1. **The differential equation.** We assume as in [Carpenter, 1998, §3.2] that we can work in the risk-neutral world even though we have non-market early termination factors. Under these assumptions we can derive the differential equation satisfied by the ESO, using the usual Black-Scholes type hedge argument. The argument is similar to that for the pricing of convertible bonds in the model of Ayache et al. [2002], Ayache et al. [2003]. We consider the pricing from the perspective of the hypothetical party that is long the ESO, for hedging, they will be short stock. The hedged portfolio in the time dt earns

$$d\Pi = r\left(V - \frac{\partial V}{\partial S}S\right)dt. \tag{1}$$

First suppose that the option has vested, it is an open period, and the option is in the money (this is exactly the situation where $\lambda(S,t) \neq 0$). Then

- with probability $1 - \lambda\, dt$ the portfolio earns $\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt$,
- with probability $\lambda\, dt$ exercise occurs. In the case of exercise, the ESO is surrendered and $S - K$ is received, for a p&l of $-V + S - K$.

Thus

$$r\left(V-\frac{\partial V}{\partial S}S\right)dt=(1-\lambda\,dt)\left(\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}\right)dt+\lambda\,dt\,(-V+S-K)$$

which simplifies (as $dt^2=0$) to

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-(r+\lambda)V+\lambda(S-K)+r\frac{\partial V}{\partial S}S=0$$

Now suppose we have the complementary situation: the option has not vested, or it is a closed period, or it is out of the money. Then

- with probability $1-\nu\,dt$ the portfolio earns $\left(\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}\right)dt$,
- with probability $\nu\,dt$ the option is forfeited. Thus the ESO is surrendered, nothing is received, and the short position in the stock is eliminated, for a p&l of $-V$.

Thus

$$r\left(V-\frac{\partial V}{\partial S}S\right)dt=(1-\nu\,dt)\left(\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}\right)dt+\nu\,dt\,(-V)$$

which simplifies (as $dt^2=0$) to

$$\frac{\partial V}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-(r+\nu)V+r\frac{\partial V}{\partial S}S=0$$

Note the important difference in the form of these two differential equations: the presence of the term $\lambda(S-K)$ in the former. But since the supports of λ, ν are complementary we can define a Black-Scholes differential operator $\mathcal{M}$ which encompasses both forms by

$$\mathcal{M}(X)=\frac{\partial X}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2 X}{\partial S^2}-(r+\lambda+\nu)X+r\frac{\partial X}{\partial S}S \tag{2}$$

Then our problem is the solution to

$$\mathcal{M}(V)+\lambda(S-K)=0. \tag{3}$$

4.2. **The finite difference equations.** In the following, the subscript i is an asset price variable (running from 0 to I) and the superscript k is a reverse time variable, starting with $k=0$ at time T to $k=K$ at valuation date t.

i is a variable which runs from $i=0$ (corresponding to a stock price of $i\Delta S=0$) to, as usual, one that corresponds to several standard deviations above the forward price of the stock i.e. I, being the top value of i, is chosen so that $I\Delta S$ is approximately to that function of the forward price, volatility, and term. One also arranges things so that there exists i_0 such that $i_0\Delta S=S$.

At large stock prices, V will be assumed linear in S, while at 0 the option is worthless. The terminal condition is clear. Thus

$$V_I^k = 2V_{I-1}^k - V_{I-2}^k \tag{4}$$

$$V_0^k = 0 \tag{5}$$

$$V_i^0 = \max(i\Delta S - K, 0) \tag{6}$$

We step backwards in time from the termination date in steps of Δt, adjusting these steps downwards when necessary to ensure that each dividend ldr date and each exercise open/close switch occurs on a node.[1] Suppose the dividend if the stock price is at $i\Delta S$ is $\alpha_i \Delta S$ (this formulation accommodates both the cash and yield setting).[2] Let $\{\cdot\}$ denote fractional part of a number. The following equation is to be understood in the following sense: all V_i^k are calculated as usual, then the given substitution is done. This cannot be done 'in place', in other words, the entire V_i^k vector needs to be calculated, the vector of results written elsewhere, and then the interpolated results below rewritten. Then

$$V_i^k = (1 - \{\alpha_i\})V^k_{\max(0, i - \lfloor \alpha_i \rfloor)} + \{\alpha_i\} V^k_{\max(0, i-1-\lfloor \alpha_i \rfloor)} \tag{7}$$

The maximum functions in the subscripts are purely to ensure that negative indices do not occur if the stock price is low.

We can use a standard Crank-Nicolson finite difference method to implement the model.[3] Let M^d be the Crank-Nicolson discretisation of $\mathcal{M}$ (see [Wilmott, 2000, Chapter 64] for example), so the system we solve is

$$0 = M^d(V) + \lambda(S - K) \tag{8}$$

When it is not an open period for exercise, this can be done directly, using a tri-diagonal solver. When it is an open period for exercise, we invoke the 'American correction' to the option price, much as in Carpenter [1998]. This is done by using a standard projective successive over-relaxation technique, as [Wilmott et al., 1995, §9.4], [Wilmott, 2000, §64.9.2].

It is important to realise that the function λ is discontinuous if there are periods where the option cannot be exercised (recall, $\lambda = 0$ in such periods). We need to manage this carefully in the finite difference scheme, for otherwise there could be material instability at in the output at those discontinuities. Suppose we have solved the finite difference scheme to time t_i, we are now solving for time t_{i+1}. (Of course time here is running

[1] This approach is permissible as the Crank-Nicolson method is not sensitive to the relative size of the time steps and stock steps.

[2] Also, if the stock price is so low as to not be able to cover a cash dividend, we will assume that whatever dividend can be paid is paid, and then bankruptcy is declared.

[3] We assume that the payoff is continuous. If the payoff is discontinuous, then the Crank-Nicolson method may demonstrate instability. But there is nothing special here that requires that scheme to be used, and the differential equation can be solved with other schemes.

backwards.) We know that the scheme is either open or closed throughout that interval, as we arranged the time grid to include all those dates where exercise is switched on or off. If the scheme is closed in that interval then we set $\lambda = 0$ in (8). On the other hand, if the scheme is open, then we set $\lambda(S, t_i)$ as in §3.

The output of such a scheme might appear as in Figure 1.

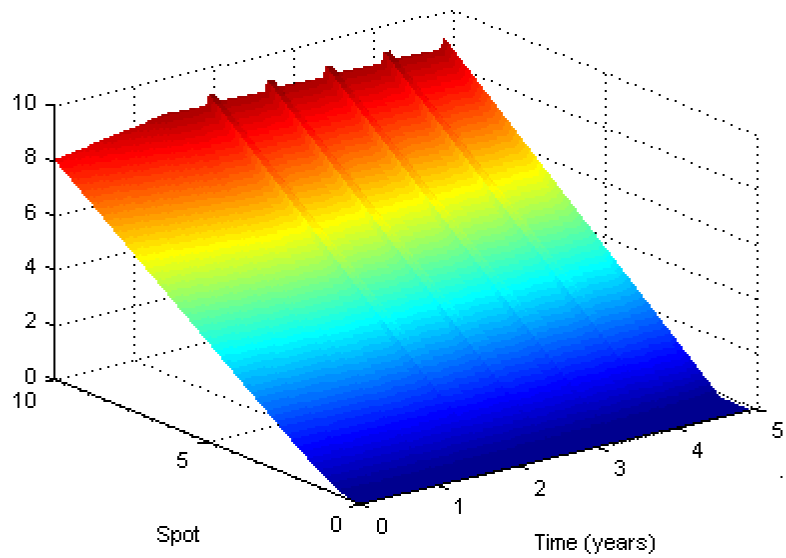

FIGURE 1. Finite difference grid valuing the option. The small discontinuities seen are due to (7).

4.3. **The rate of early exercise.** When expressed in a hazard rate formulation, many choices for choosing the functional form of $\lambda(S, t)$, and calibration thereof, are possible. There exists a significant literate on this subject, and it is our intention to be users of, not contributors to, this debate. We note that the formulation we establish here can be used with many early exercise probability/hazard functions - either parametric or empirical - except for those which rely on some path-dependent information. Thus, this hazard rate formulation can be used with any Markovian factors. See Klein and Maug [October 14, 2008] where an extensive statistical analysis of the significance of many (both Markovian and non-Markovian) factors is undertaken.

As an example, the Hull-White model Hull and White [2004], fits into this framework: one sets $\lambda(S, t) = 0$ if S is below their 'early exercise multiple M', and $\lambda(S, t) = \infty$ if above.[4]

The choice of toy function we have made in §4.4 is as follows: $P(S, t)$ is the probability of a down-and-out put expiring in the money, where the barrier of the down-and-out is the strike of the ESO, the strike of the down-and-out B is a free parameter, and all other variables have their usual meaning. The choice of B is calibrated using the history of early exercise in the scheme, or to such in peer schemes. The relevant probability can then

[4]In a numerical scheme one sets $\lambda(S, t)$ to be some very large number.

be deduced from vanilla barrier or rebate option pricing formulae. Thus if $S > K$ and we are in an open period then [Musiela and Rutkowski, 1998, §9.6]

$$P(S,t) = N\left(\frac{-\ln\frac{B}{S} + m_-(T-t)}{\sigma\sqrt{T-t}}\right) - \left(\frac{K}{S}\right)^{2m_-/\sigma^2} N\left(\frac{-\ln\frac{B}{S} + 2\ln\frac{K}{S} + m_-(T-t)}{\sigma\sqrt{T-t}}\right) \tag{9}$$

where $m_- = r - q - \frac{1}{2}\sigma^2$. Given the exercise probability $P(S,t)$ we put, as is standard in hazard rate calculations,

$$e^{-\lambda(S,t)(T-t)} = 1 - P(S,t).$$

This function indeed satisfies the minimum properties identified earlier. This is simply one possibilities. It is convenient as a parametric form that satisfies the requisite requirements of any early exercise function.

In order to avoid explosion of this function as the term outstanding descends to 0, we set a lower bound to $T - t$ to be 1. The annualised rates of early exercise one gets are as in Figure 2. At any given time, the lower the stock price, the less likely the employee is to exercise value.

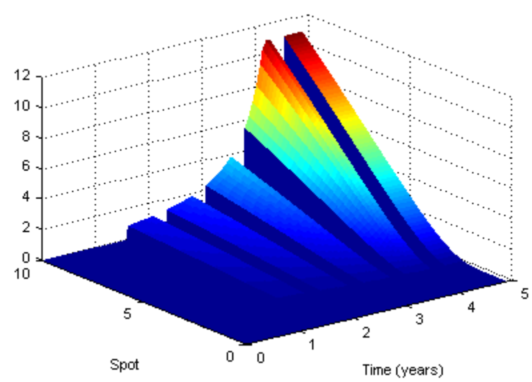

Figure 2. Modelled hazard rates for varying stock price, for a typical ESO scheme with five distinct open periods.

We have a very simple parametric model which creates this λ function. There is one new parameter: this is the ratio $p = \frac{B}{S}$, which we have simply named the hazard calibration parameter. As p increases, the λ function decreases, and the option value increases. In the limit as $p \uparrow \infty$ one has the risk-neutral world where there is no tendency to exercise early. Empirically we find that a value of $p = 5$ or 6 suffices to get an option value very close to that value. On the other hand, as p decreases, the value decreases, as the tendency to exercise early increases. See Figure 3.

4.4. **Option pricing examples.** We can compare the impact of forfeiture using our model against the naïve model of post-multiplying the value of the employee stock options found by assuming no attrition with the survival fraction. The impact of attrition is not linear, for example, attrition is less likely when the option is well in the money and/or near to vesting than when not. See Figure 4.

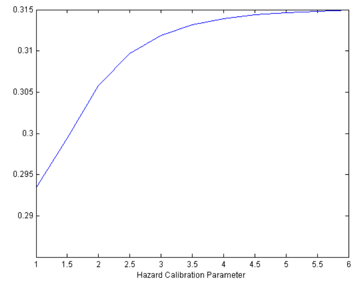

FIGURE 3. Lower values of the hazard calibration parameter correspond to greater willingness to exercise early, so lower option value.

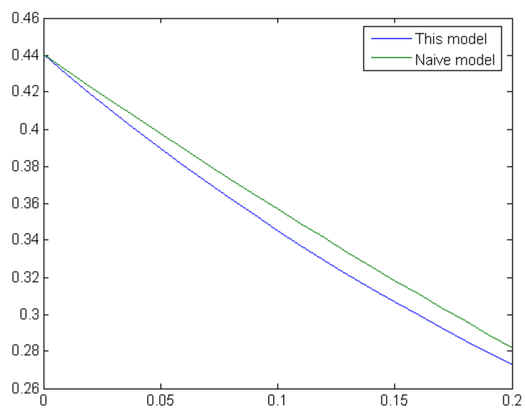

FIGURE 4. Higher forfeiture obviously lowers option value.

We can experience explicit instances of how pricing in this world is different to pricing in the risk-neutral world. In a risk neutral world, as we move along the continuum from European to American options, widening the various Bermudan windows, the option price steadily increases. Here it is possible to observe the phenomenon of decreases in the liability for the firm. As the Bermudan windows widen, the option holder is more and more likely to exercise sub-optimally, and the liability can decrease. Having closed periods forces the option holder to defer exercise, and this may be forcing them to behave more optimally from a risk-neutral point of view.

See Figure 5. In this example, interfering factors - forfeitures, and dividends after vesting - are 'switched off' i.e. set to 0. The only factor additional to purely vanilla considerations is the hazard rate λ. We consider a six year open, expiring on the 31 Dec of the sixth year, vesting on the 1 Jan of the fourth year. We consider exercise windows starting on the first of each of the 12 months of the year, in each of the fourth, fifth and sixth years. The exercise window always ends on the 31 Dec of that year. The horizontal axis is the month of the year where the window starts. As the exercise window becomes narrower, the value of the option decreases. In a fairly simple at-the-money example, the liability can increase by as much as 30%, say.

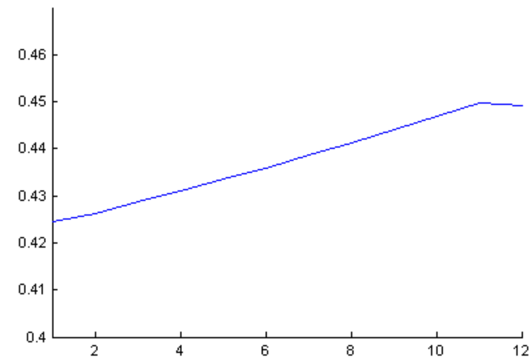

FIGURE 5. On the horizontal axis, the months of the year in which the open exercise period may commence. As the window gets narrower, the value increases, because opportunities for 'suboptimal' behaviour are reduced.

If we switch off the λ function too - this is achieved by setting the hazard calibration parameter to a very large number, which makes $\lambda = 0$ - then the value of the option for each of the window openings is, to the accuracy of the finite difference scheme, the same. This is because we have reduced the option to a vanilla Bermudan call option with no dividends.

Similarly, the value of the option does not necessarily increase with increasing volatility, although this effect is less marked. For low volatilities, the probability of large increases in stock price is small, and hence the probability of early exercise is small (the λ function is small). As volatility increases, early exercises become more prevalent, and this 'suboptimal' behaviour can cause a decrease in value. But at some point the increase in value in the usual sense of the word (the increased probability of large unexercised intrinsic value) starts to dominate this value-decreasing factor.

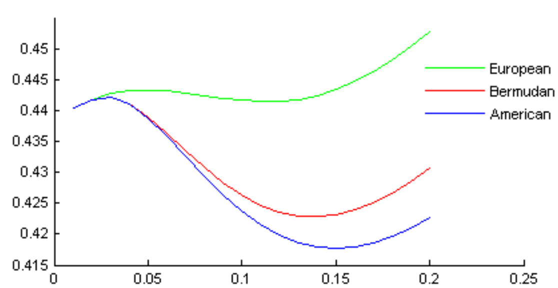

FIGURE 6. Option value as a function of volatility. The American option vests immediately, the Bermudan option vests half way to expiry, the European option vests as it expires.

Bibliography

E. Ayache, P.A. Forsyth, and K.R. Vetzal. Next generation models for convertible bonds with credit risk. *WILMOTT Magazine*, November:68–77, 2002. 5

E. Ayache, P.A. Forsyth, and K.R. Vetzal. Valuation of convertible bonds with credit risk. *Journal of Derivatives*, 11(Fall):9–30, 2003. 5

Zvi Bodie, Robert S. Kaplan, and Robert C. Merton. For the last time: Stock options are an expense. *Harvard Business Review*, 2003. 1

Jennifer N. Carpenter. The exercise and valuation of executive stock options. *Journal of Financial Economics*, 48:127–158, 1998. 1, 2, 4, 5, 7

John Cox, Stephen Ross, and Mark Rubinstein. Option pricing: a simplified approach. *Journal of Financial Economics*, 7:229–263, 1979. 2, 4

J. C. Hull and A. White. How to value employee stock options. *Financial Analysts Journal*, 60(1):114–119, 2004. 8

IASCF. International financial reporting standard 2: Share-based payment, 2006. This version includes amendments resulting from IFRSs issued up to 31 December 2005. 1, 2, 5

Daniel Klein and Ernst Maug. How do executives exercise stock options?, October 14, 2008. URL http://cbe.anu.edu.au/schools/FAS/documents/Maug_17Oct2008.pdf. 5, 8

Marek Musiela and Marek Rutkowski. *Martingale Methods in Financial Modelling*, volume 36 of *Applications of Mathematics: Stochastic Modelling and Applied Probability*. Springer, 1998. 9

Paul Wilmott. *Paul Wilmott on Quantitative Finance*. John Wiley and Sons Ltd., 2000. 7

Paul Wilmott, Sam Howison, and Jeff Dewynne. *The Mathematics of Financial Derivatives: A Student Introduction*. Cambridge University Press, 1995. 7

Financial Modelling Agency, 19 First Ave East, Parktown North, 2193, South Africa.

E-mail address: graeme@finmod.co.za